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5.4 Wave Motion in Classical Physics
Waves: propagation of energy, not particles

- Pressure
- Surface height
- Electric and magnetic fields
- Rope height

\[ \psi(x, t) \]
Different Waveforms:

- A pulse
- A wave train
- A continuous harmonic wave
Two Snapshots of a Wave Pulse

propagating with velocity $v$ along the $x$-axis

At $t = 0$, the peak is at $x = 0$.

At $t$, the peak is at $x = vt$.

$t = 0$

$\psi(x, t = 0) = f(x)$

$t \geq 0$

$\psi(x, t) = f(x - vt)$
Propagation towards Positive x-direction

\[ \psi(x, t) = f(x - vt) \quad \text{if} \quad v > 0 \]

Propagation towards Negative x-direction

\[ \psi(x, t) = f(x + vt) \quad \text{if} \quad v > 0 \]
Fingerprint of the Wave Phenomena: $x - vt$

$$\psi(x, t) = f(x - vt)$$

$$\psi(x, t = 0) = A e^{-\frac{x^2}{\sigma^2}}$$

$$\psi(x, t) = A e^{-\frac{(x-vt)^2}{\sigma^2}}$$

$$\psi(x, t = 0) = A e^{-\frac{|x|}{\sigma}}$$

$$\psi(x, t) = A e^{-\frac{|x-vt|}{\sigma}}$$

$$\psi(x, t = 0) = A \cos[kx]$$

$$\psi(x, t) = A \cos[k(x-vt)]$$
\[ \psi(x, t) = f(x - \nu t) \]

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 \psi}{\partial t^2}
\]

Appendix 1
Wave Equation: \[
\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}
\]

- 2\textsuperscript{nd} order partial differential equation
- Linear equation in \(\psi\): \(\psi\), \(\partial_x \psi\), \(\partial_x^2 \psi\), ...
- Homogeneous equation: no term involving independent variables

If \(\psi_1\) and \(\psi_2\) are solutions then a linear combination \(\alpha \psi_1 + \beta \psi_2\) is also a solution equation.
Harmonic Wave Solution:

\[ \psi(x, t) = A \cos[k(x - vt) + \varepsilon] \]

or

\[ \psi(x, t) = B \sin[k(x - vt) + \varepsilon'] \]

or

\[ \psi(x, t) = A \cos[k(x - vt) + \varepsilon] + B \sin[k(x - vt) + \varepsilon'] \]

or

\[ \psi(x, t) = \text{Re} \left\{ A e^{i[k(x - vt) + \varepsilon]} \right\} \]
Wavelength in a Harmonic Wave:

\[ \psi(x, t) = A \cos[k(x - v t) + \varepsilon] \]

The wavelength can be measured between any two repeating points on the wave.

\[ t = \text{fixed} \]

\[ k \equiv \frac{2 \pi}{\lambda} \]
Period in a Harmonic Wave:

\[ \psi(x, t) = A \cos[k(x - v t) + \varepsilon] \]

\( x = \text{fixed} \)

\[ k \nu \equiv \frac{2 \pi}{\tau} \quad \text{or} \quad \tau \equiv \frac{\lambda}{\nu} \]
Wave Speed

\[ v = \frac{\lambda}{\tau} \]
A Few Definitions:

\[ \psi(x, t) = A \cos[k(x - \nu t) + \varepsilon] \]

\[ \nu = \frac{\lambda}{\tau} \]

Wave Number

\[ k \equiv \frac{2 \pi}{\lambda} \]

Frequency

\[ f \equiv \frac{1}{\tau} \]

Angular Frequency

\[ \omega \equiv \frac{2 \pi}{\tau} = 2 \pi f \]

\[ \psi(x, t) = A \cos(k x - \omega t + \varepsilon) \]

\[ \nu = \frac{\omega}{k} \]
\[ \psi(x, t) = A \cos(kx - \omega t + \epsilon) \]
\[ = A \cos[\varphi(x, t)] \]

**Amplitude:** \( A \)

**Phase:** \( \varphi(x, t) \equiv kx - \omega t + \epsilon \)
Constant Phase and Phase Velocity

\[ \varphi(x, t) \equiv k x - \omega t + \varepsilon \]

\[ d\varphi(x, t) = k \, dx - \omega \, dt \]

\[ d\varphi(x, t) = 0 \quad \Rightarrow \quad k \, dx - \omega \, dt = 0 \]

\[ v_{ph} = \frac{dx}{dt} = \frac{\omega}{k} = \frac{\lambda}{\tau} \]
Adding Two Waves with Same Wavelength and Frequency:
(Almost Completely) In Phase:

\[ \psi_1 = \sin(\theta) \quad \psi_2 = \sin(\theta + \frac{\pi}{20}) \]

Constructive Interference
(Almost Completely) Out of Phase:

\[ \psi_1 = \sin(\theta) \quad \psi_2 = \sin\left(\theta + \frac{\pi}{20} + \pi\right) \]

Destructive Interference
Interference
Wave Propagation Through Slits:

One slit:

\[ \psi = \frac{\psi_0 e^{i(kr - \omega t)}}{r} \]

Two slits:

\[ \psi = \psi_1 + \psi_2 \]

\[ \psi_1 = \frac{\psi_{01} e^{i(kr_1 - \omega t)}}{r_1} \]

\[ \psi_2 = \frac{\psi_{02} e^{i(kr_2 - \omega t)}}{r_2} \]

\[ I \equiv \langle |\psi|^2 \rangle \]
Interference of Waves:

\[ \text{Intensity} = \frac{\text{power}}{\text{area}} = I(\mathbf{r}) \equiv \langle |\psi(\mathbf{r}, t)|^2 \rangle \]
Phase Difference in Young's Interferometer

\[ r_1 - r_2 \approx a \sin(\theta) = a \frac{Y}{R} \]

\[ \Lambda = \frac{\lambda R}{a} \]

\[ \psi(\vec{r}, t) = \psi_1(\vec{r}, t) + \psi_2(\vec{r}, t) \]

\[ I(\vec{r}) = \langle |\psi(\vec{r}, t)|^2 \rangle = I_1 + I_2 + 2 \sqrt{I_1 I_2} \cos[k(r_1 - r_2) + (\varepsilon_1 - \varepsilon_2)] \]
Interference:

\[ \psi = \psi_1 + \psi_2 + \cdots = \sum_i \psi_i \]

Diffraction:

\[ \psi = \int \int \psi_i \, da_i \text{ over aperture} \]
\[ \psi(\vec{r}, t) = \iint_{\text{aperture}} \frac{\psi_0 e^{i(kr - \omega t)}}{r} \, dy \, dz \]
Single Slit

\[ r = \sqrt{s^2 + (Y-y)^2} = \sqrt{R^2 + y^2 - 2yY} \approx R - \frac{yY}{R} \]

\[ \psi(Y, t) = \frac{\psi_0 e^{i \{k R - \omega t\}}}{R} \int_{-d/2}^{+d/2} e^{-i \frac{kY}{R} y} dy = \frac{\psi_0 e^{i \{k R - \omega t\}}}{R} \sin \left( \frac{kYd}{2R} \right) \]

Incident light waves

\[ r = \sqrt{s^2 + (Y-y)^2} \]

\[ \theta \]

\[ Y \]

\[ d \]

\[ s \]

\[ R \]

\[ \psi(Y, t) = \frac{\psi_0 e^{i \{k R - \omega t\}}}{R} \int_{-d/2}^{+d/2} e^{-i \frac{kY}{R} y} dy = \frac{\psi_0 e^{i \{k R - \omega t\}}}{R} \sin \left( \frac{kYd}{2R} \right) \]
**Single Slit, cont.**

\[ I(Y) = \langle |\psi|^2 \rangle = I_0 \left[ \frac{\sin \left( \frac{k Y d}{2 R} \right)}{\frac{k Y d}{2 R}} \right]^2 \]

zeros at \[ \frac{k Y_m d}{2 R} = m \pi \]

with \[ m = \pm 1, \pm 2, \pm 3 \]

\[ Y_m = m \frac{\lambda R}{d} \]

\[ \frac{Y_m}{R} = \sin(\theta_m) = m \frac{\lambda}{d} \]

\[ \frac{I(Y)}{I_0} \]

\[ \sin(\theta_1) = \frac{\lambda}{d} \]
After diffraction, the wave spreads over a range of angles:

\[ \Delta \{ \sin(\theta) \} \approx \Delta \theta \geq \frac{\lambda}{d} \]
PhET
Interference

\[ \psi(\vec{r}, t) = \psi_1(\vec{r}, t) + \psi_2(\vec{r}, t) \]

\[ \langle |\psi(\vec{r}, t)|^2 \rangle \equiv I = I_1 + I_2 + 2 \sqrt{I_1 I_2} \cos \left[ 2 \pi \frac{a Y}{\lambda R} + (\varepsilon_1 - \varepsilon_2) \right] \]

\[ \Lambda = \frac{\lambda R}{a} \]
\[
\Delta \theta \geq \frac{\lambda}{d}
\]

\[
\psi(\vec{r}, t) = \iiint_{\text{aperture}} \frac{\psi_0 e^{i(k r - \omega t)}}{r} \, dy \, dz
\]

\[
\langle |\psi|^2 \rangle \equiv I = I_0 \left[ \frac{\sin \left( \frac{\pi Y d}{\lambda R} \right)}{\left( \frac{\pi Y d}{\lambda R} \right)} \right]^2
\]
Adding Two Waves of Different Frequencies (therefore, Different Wavelengths)
Adding Two Waves:

$$\psi(x, t) = 2 \psi_0 \cos[k x - \omega t + \varepsilon] \cos[x \Delta k - t \Delta \omega + \Delta \varepsilon]$$

**Appendix 2**

$$\psi_{0,1} = \psi_{0,2} = \psi_0$$

\[ k \equiv \frac{1}{2}(k_1 + k_2) \]
\[ \Delta k \equiv \frac{1}{2}(k_1 - k_2) \]
\[ \omega \equiv \frac{1}{2}(\omega_1 + \omega_2) \]
\[ \Delta \omega \equiv \frac{1}{2}(\omega_1 - \omega_2) \]
\[ \varepsilon \equiv \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \]
\[ \Delta \varepsilon \equiv \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \]
\[ \psi(x, t) = 2 \psi_0 \cos[k x - \omega t + \varepsilon] \cos[x \Delta k - t \Delta \omega + \Delta \varepsilon] \]

\[ k x - \omega t + \varepsilon = \varphi \text{ phase velocity} \]

\[ x \Delta k - t \Delta \omega + \Delta \varepsilon = \varphi_m \text{ group velocity} \]

\[ v_{ph} = \frac{\omega}{k} \]

\[ v_{gr} = \frac{d\omega}{dk} \]
Fourier Theory
Harmonic Function

\[ \psi(t) = A e^{i(kz - \omega t)} \]

One harmonic function extends all the way from \(-\infty\) to \(+\infty\).

Can we use Harmonic Functions to build a Spatially Confined Function?
Fourier Analysis

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i \omega t} d\omega \]

\[ F(\omega) = \int_{-\infty}^{\infty} f(t') e^{+i \omega t'} dt' \]

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{-i k_x x} dk_x \]

\[ F(k_x) = \int_{-\infty}^{\infty} f(x') e^{+i k_x x'} dx' \]
Example 1: Top Hat Function

\[ f(t) = \begin{cases} A & \text{when } -\frac{T}{2} \leq t \leq +\frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ f(t) = \begin{cases} A & \frac{t}{2} \\ 0 & \text{otherwise} \end{cases} \]
\[ F(\omega) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} A e^{i \omega t'} dt' \]

\[ F(\omega) = A T \text{sinc} \left( \frac{\omega T}{2} \right) \]

\[ F(\omega) = 0 \quad \text{at} \quad \frac{\omega_m T}{2} = m \pi \]

\[ m = \pm 1, \pm 2, \pm 3, \ldots \]

\[ \omega_{+1} - \omega_{-1} = \frac{4 \pi}{T} \quad \Rightarrow \quad T \Delta \omega \approx 4 \pi \]
\[ f(t) = \begin{cases} A & \text{when } -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ F(\omega) = AT \text{sinc}\left(\frac{\omega T}{2}\right) \]

An even function

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} AT \text{sinc}\left(\frac{\omega T}{2}\right) e^{-i\omega t} \, d\omega \]

\[ = \frac{1}{\pi} \int_{0}^{\infty} AT \text{sinc}\left(\frac{\omega T}{2}\right) \cos(\omega t) \, d\omega \]
Truncated Cosine Function:

\[ f(t) = \begin{cases} 
A \cos(\omega_0 t) & \text{when } -\frac{T}{2} \leq t \leq \frac{T}{2} \\
0 & \text{otherwise} 
\end{cases} \]

\[ \Delta t \Delta \omega \approx \pi \]
Gaussian Function:

\[ f(x) = e^{-\Delta k^2 x^2} \cos(k_0 x) \]

\[ \Delta x \Delta k_x \approx \frac{1}{2} \]
In Summary, Fourier Theory gives us:
A wave of finite time duration can never be monochromatic!!

\[ \psi(z, t) = A e^{i(k z - \omega t)} \]

\[ \Delta t \Delta \omega \geq \frac{1}{2} \]

\[ \psi(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i(k z - \omega t)} d\omega \]
Lateral confinement of a wave leads to divergence!!

\[ k = \frac{2\pi}{\lambda} \hat{e}_z \]

\[ \psi(z, t) = A e^{i(\omega z - \omega t)} \]

\[ \Delta x \Delta k_x \geq \frac{1}{2} \]

\[ \psi(x, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{i(k_x x + k_z z - \omega t)} \, dk_x \]

\[ (k_x)^2 + (k_z)^2 = k^2 = \left(\frac{2\pi}{\lambda}\right)^2 \]
5.1 X-Ray Scattering

- Max von Laue suggested that if x rays were a form of electromagnetic radiation, interference effects should be observed.

- Crystals act as three-dimensional gratings, scattering the waves and producing observable interference effects.
X-Ray Scattering and Bragg's Law

- William Lawrence Bragg interpreted the x-ray scattering as the re-emission of the incident x-ray beam from a unique set of planes of atoms within the crystal.
Bragg’s Law

- There are two conditions for constructive interference of the scattered x-rays by a crystal:

1) The angle of incidence must equal the angle of reflection of the outgoing wave.

2) The difference in path lengths must be an integral number of wavelengths:

\[ n \lambda = 2d \sin(\theta) \]

\[ n = 1, 2, 3, \ldots \]
\[ n \lambda = 2d \sin(\theta) \]
Rosalind Franklin produced the X-ray diffraction images of the DNA molecule that helped Watson and Crick unravel the DNA structure.
X-Ray Diffraction Nowadays

Crystal → Diffraction pattern → Electron density map → Protein model
5.2 de Broglie Waves, 1920
For Electromagnetic Waves:

- According to Special Relativity, and \( m = 0 \): \( E = p \ c \)

- According to Planck, Einstein, Compton, etc: \( E = hf \)

\[
E = pc = hf \\
p = \frac{hf}{c} = \frac{h}{\lambda}
\]
de Broglie Hypothesis:

- Maybe matter also behaves like waves.

- With the wavelength given by:

\[
\lambda = \frac{\hbar}{p}
\]

\[
p = \frac{1}{c} \sqrt{(E_{\text{total}})^2 - (mc^2)^2}
\]

\[
\approx \sqrt{2mK} \quad \text{when} \quad K \ll mc^2
\]
Example 5.2

Tennis ball

\[ m = 57 \text{ g} \]
\[ v = 25 \text{ m/s} \quad (56 \text{ mph}) \]
\[ \lambda = \frac{h}{p} \approx \frac{h}{\sqrt{2 m K}} \]
\[ = \frac{6.63 \times 10^{-34} \text{ J s}}{57 \times 10^{-3} \text{ kg} \ 25 \text{ m/s}} \]
\[ = 4.7 \times 10^{-34} \text{ m} \]

Electron

\[ K = 50 \text{ eV} \]
\[ \lambda = \frac{h}{p} \approx \frac{h}{\sqrt{2 m K}} \]
\[ = \frac{h c}{\sqrt{2 m c^2 K}} \]
\[ = \frac{1240 \text{ eV nm}}{\sqrt{2 \times 0.511 \times 10^6 \text{ eV} \ 50 \text{ eV}}} \]
\[ = 0.17 \text{ nm} \]
de Broglie Wavelength

\[ \lambda = \frac{h}{p} \]

\[ p = \frac{1}{c} \sqrt{(E_{\text{total}})^2 - (mc^2)^2} \]

\[ = \frac{E_{\text{total}}}{c} = \frac{hf}{c} \quad \text{when} \quad m = 0 \quad (\text{e.g., photons}) \]

\[ \cong \sqrt{2mK} \quad \text{when} \quad K \ll mc^2 \quad (\text{e.g., non-relativistic particles}) \]
Clinton J. Davisson (1881–1958) is shown here in 1928 (right) looking at the electronic diffraction tube held by Lester H. Germer (1896–1971). Davisson received his undergraduate degree at the University of Chicago and his doctorate at Princeton. They performed their work at Bell Telephone Laboratory located in New York City. Davisson received the Nobel Prize in Physics in 1937.
Electron Scattering
Intensity = radial distance along dashed line to data at angle $\phi$

- $\phi$
- Data
- Peak
- $50^\circ$

- 44 eV
- 48 eV
- 54 eV
- 64 eV
- 68 eV
Transmission Electron Microscope
diffraction photographs

120-keV electrons scattered on the quasicrystal $\text{Al}_{80}\text{Mn}_{20}$.

Electron diffraction pattern on beryllium.

The dots in (a) indicate that the sample was a crystal, whereas the rings in (b) indicate that a randomly oriented sample (or powder) was used.
Scanning Electron Microscope
Combining Bohr’s Quantization of the Angular Momentum $L = n \hbar$ & de Broglie Matter Waves $p = \frac{h}{\lambda}$

\[
\frac{h}{\lambda} \cdot r = \rho r = L = n \hbar
\]

\[
2 \pi r = n \lambda
\]
5.5 Wave or Particle?
Interference with Wave-Particle:
Young's Interference with the Photon Wave-Particle:

\[ \psi = \psi_1 + \psi_2 \]

\[ |\psi|^2 \]
Young's Interference with the Electron Wave-Particle:

Demonstration of electron interference using two slits similar in concept to Young's double-slit experiment for light.

The result by Claus Jönsson (1961) clearly shows that electrons exhibit wave behavior.
Experimental Facts:

- The interference pattern only appears when both slits are opened.
- The wave-particle (photon, electron, ...) cannot be divided.
- One wave-particle passes through only one slit.
- An experiment to locate in which slit the wave-particle passed through destroys the interference pattern.
Copenhagen Interpretation
There is a wave-function $\psi$ associated with each possible path allowed for the wave-particle.

The $|\psi|^2$ determines the probability of finding a particle at a given place and time.

The average number of wave-particles $N$ is proportional to $|\psi|^2$. It's a statistical value, therefore, non-deterministic.

The interference pattern comes from the interference of the individual wave-functions associated with each path: $|\psi_1 + \psi_2|^2$. 
Diffraction with Wave-Particles:

The wavelength $\lambda$ of the wave-function $\psi$ is determined by the de Broglie relation: $\lambda = \frac{h}{p}$. 
After diffraction, the wave-function spreads over a range of angles:

\[ \Delta \{ \sin(\theta) \} \cong \Delta \theta \geq \frac{\lambda}{L} \]

\[ \frac{L}{\lambda} \Delta \theta \geq 1 \]

\[ L \Delta p = L p \Delta \theta = L \frac{h}{\lambda} \Delta \theta \geq h \]

\[ \Delta x \Delta p_x \geq h \]
There is a fundamental limitation in the simultaneous knowledge of position and linear momentum.

\[ L = \Delta x \]

\[ \Delta x \Delta p_x \geq h \]
When we attempt to find the position of an electron:

\[ p = \frac{h}{\lambda} \]

\[ \Delta p_x = 2 \, p \, \sin(\theta) \]
\[ = 2 \frac{h}{\lambda} \sin(\theta) \]

\[ \Delta x = \frac{\lambda}{\sin(\theta')} \]

\[ \Delta x \, \Delta p_x \approx 2 \, h \]
5.6 Heisenberg Uncertainty Principle

\[ \Delta x \Delta k_x \geq \frac{1}{2} \]

\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \]

\[ \Delta t \Delta \omega \geq \frac{1}{2} \]

\[ \Delta t \Delta E \geq \frac{\hbar}{2} \]
In Classical Physics:

- The position and linear momentum of a particle can be known simultaneously with absolute precision.

- The Classical Laws (e.g., Newton's Law) unambiguously describe (deterministically) the evolution of a particular system.

- The combination of deterministic laws with the absolute knowledge of the initial conditions (e.g., position and linear momentum) allows one to completely and precisely predict the future.
In Quantum Physics:

- There is a fundamental limitation on the simultaneous knowledge (through measurement) of a particular experimental variable and its associated (conjugated) variable:

\[
\Delta x \, \Delta p_x \geq \frac{\hbar}{2}
\]

\[
\Delta t \, \Delta E \geq \frac{\hbar}{2}
\]
Consequences:

\[ \Delta p_x \geq \frac{\hbar}{2 \Delta x} \]

(Non-Relativistic) Minimum Kinetic Energy

\[ K = \frac{p_x^2}{2 m} \approx \frac{(\Delta p_x)^2}{2 m} \geq \frac{\hbar^2}{8 m \Delta x^2} \]
5.7 Wave Functions and Probability
For the **Photon** Wave-Particle:

\[
\frac{\text{energy}}{\text{unit area} \times \text{unit time}} = I = \frac{\text{number of photons}}{\text{unit area} \times \text{unit time}} \times h f
\]

\[
\epsilon_o c |E|^2 = I = N hf
\]

\[
N = \frac{\text{number of photons}}{\text{unit area} \times \text{unit time}}
\]

\[
|E|^2 \propto N \propto \text{Probability}
\]
For Any Wave-Particle:

\[ |\psi|^2 \propto Probability \]

- The probability describes the likelihood of occurrence of an event, but it does not guarantee its occurrence.

- By using the Wave-Function, \( \psi(x, t) \), we are adopting a non-deterministic description of nature.
5.8 Wave-Particle in a Box

\[ k \ell = n \pi \Rightarrow k = n \frac{\pi}{\ell} \]

**Linear Momentum**

\[ p_n = \frac{\hbar}{\lambda} = \hbar k = n \frac{\hbar \pi}{\ell} \]

**Kinetic Energy**

\[ K_n = \frac{p_n^2}{2m} = n^2 \frac{\hbar^2}{8m\ell^2} \]

\[ \psi(x) = A \sin(kx) \]

\[ |\psi(x)|^2 = |A|^2 \sin^2(kx) \]
Appendix 1
\[ \psi(x, t) = f(x - v t) \equiv f(\alpha) \quad \alpha \equiv x - v t \]

\[ \frac{\partial \psi}{\partial x} = \frac{\partial \alpha}{\partial x} \frac{\partial \psi}{\partial \alpha} = 1 \frac{\partial \psi}{\partial \alpha} = \frac{df}{d\alpha} \]

\[ \frac{\partial \psi}{\partial t} = \frac{\partial \alpha}{\partial t} \frac{\partial \psi}{\partial \alpha} = (-v) \frac{\partial \psi}{\partial \alpha} = -v \frac{df}{d\alpha} \]

\[ \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) = \frac{\partial \alpha}{\partial x} \frac{\partial}{\partial \alpha} \left( \frac{df}{d\alpha} \right) = 1 \frac{d}{d\alpha} \left( \frac{df}{d\alpha} \right) = \frac{d^2 f}{dx^2} \]

\[ \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial t} \right) = \frac{\partial \alpha}{\partial t} \frac{\partial}{\partial \alpha} \left( -v \frac{df}{d\alpha} \right) = (-v) \frac{d}{d\alpha} \left( -v \frac{df}{d\alpha} \right) = v^2 \frac{d^2 f}{dx^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \]

\[ \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]
Appendix 2
Two waves with different frequencies (and wavelengths):

\[ \psi_1(x,t) = \psi_{0,1} \cos(k_1 x - \omega_1 t + \varepsilon_1) \]

\[ \psi_2(x,t) = \psi_{0,2} \cos(k_2 x - \omega_2 t + \varepsilon_2) \]

\[ \psi(x,t) = \psi_1(x,t) + \psi_2(x,t) \]
\[
A \equiv \frac{1}{2} (k_1 + k_2) x - \frac{1}{2} (\omega_1 + \omega_2) t + \frac{1}{2} (\varepsilon_1 + \varepsilon_2)
\]

\[
B \equiv \frac{1}{2} (k_1 - k_2) x - \frac{1}{2} (\omega_1 - \omega_2) t + \frac{1}{2} (\varepsilon_1 - \varepsilon_2)
\]

\[
a \equiv \frac{1}{2} (\psi_{0,1} + \psi_{0,2})
\]

\[
b \equiv \frac{1}{2} (\psi_{0,1} - \psi_{0,2})
\]

\[
\psi_1(x, t) = (a + b) \cos(A + B)
\]

\[
\psi_2(x, t) = (a - b) \cos(A - B)
\]
\[ \psi(x, t) = \psi_1(x, t) + \psi_2(x, t) \]

\[ = (a + b) \cos(A + B) + (a - b) \cos(A - B) \]

\[ = a \left[ \cos(A + B) + \cos(A - B) \right] + b \left[ \cos(A + B) - \cos(A - B) \right] \]

\[ = 2a \cos(A) \cos(B) - 2b \sin(A) \sin(B) \]
Same Amplitude

\[ \psi_{0,1} = \psi_{0,2} = \psi_0 \]

\[ a \equiv \frac{1}{2} (\psi_{0,1} + \psi_{0,2}) = \psi_0 \]

\[ b \equiv \frac{1}{2} (\psi_{0,1} - \psi_{0,2}) = 0 \]

\[ \psi(x, t) = 2 \ a \ \cos(A) \ \cos(B) \]

\[ = 2 \ \psi_0 \ \cos \left[ \frac{1}{2} (k_1 + k_2) x - \frac{1}{2} (\omega_1 + \omega_2) t + \frac{1}{2} (\varepsilon_1 + \varepsilon_2) \right] \]

\[ \cos \left[ \frac{1}{2} (k_1 - k_2) x - \frac{1}{2} (\omega_1 - \omega_2) t + \frac{1}{2} (\varepsilon_1 - \varepsilon_2) \right] \]
\[ k \equiv \frac{1}{2}(k_1 + k_2) \]
\[ \Delta k \equiv \frac{1}{2}(k_1 - k_2) \]
\[ \omega \equiv \frac{1}{2}(\omega_1 + \omega_2) \]
\[ \Delta \omega \equiv \frac{1}{2}(\omega_1 - \omega_2) \]
\[ \epsilon \equiv \frac{1}{2}(\epsilon_1 + \epsilon_2) \]
\[ \Delta \epsilon \equiv \frac{1}{2}(\epsilon_1 - \epsilon_2) \]

\[ \psi_{0,1} = \psi_{0,2} = \psi_0 \]

\[ \psi(x,t) = 2 \psi_0 \cos[kx - \omega t + \epsilon] \cos[x \Delta k - t \Delta \omega + \Delta \epsilon] \]
Appendix 3
Group Velocity of Wave-Particels
**Group Velocity: non-relativistic case**

\[ v_g = \frac{d\omega}{dk} = \frac{d(\hbar \omega)}{d(\hbar k)} = \frac{dE}{dp} \approx \frac{dE_{nr}}{dp} = \frac{p}{m} \]

\[ E = h f = \hbar \omega \]

\[ p = \frac{h}{\lambda} = \hbar k \]

\[ E = K + m c^2 \]

\[ E_{nr} \approx \frac{p^2}{2m} + m c^2 \]
Group Velocity: general case

\[ v_g = \frac{d\omega}{dk} = \frac{d(\hbar \omega)}{d(\hbar k)} = \frac{dE}{dp} = \frac{p \, c^2}{E} = \beta \, c \]

\[ E^2 = p^2 \, c^2 + m^2 \, c^4 = \gamma^2 \, m^2 \, c^4 \]

\[ 2 \, E \, dE = 2 \, p \, c^2 \, dp \]

\[ 1 = \frac{p^2 \, c^2}{E^2} + \frac{1}{\gamma^2} \quad \Rightarrow \quad 1 - \frac{1}{\gamma^2} = \frac{p^2 \, c^2}{E^2} \quad \Rightarrow \quad \beta^2 = \frac{p^2 \, c^2}{E^2} \]