The Nuclear Atom

\[ \text{atom} \approx 10^{10} \text{ m} = 1 \text{ Å} = 0.1 \text{ nm} \]

\[ m_e < m_{\text{atom}} \]

\[ Q_{\text{atom}} \approx \text{neutral} \]

+ J. Thompson’s atomic model:

- electrons distributed over a fluid-like volume containing
  most of the mass and the positive charges

+ Rutherford’s experiment and atomic model (1911):

- \( \alpha \)-particles: doubly ionized He-atom

- some \( \alpha \)-particles undergo large scattering angles

- large scattering angles are not expected with Thompson’s model

- nucleus model: most positive charge and mass are concentrated
  in a small fraction of the whole atom
• Scattering of an α-particle by an atomic nucleus

\[ \frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{D}{2b^2} (\cos \varphi - 1) \]

\[ D = \frac{K q_x e}{m \omega \frac{v^2}{2}} \]

\[ K = \frac{1}{4\pi \varepsilon_0} \]

hyperbola in polar coordinates

\[ b : \text{impact parameter} \]

\[ r \to \infty \implies \varphi + \theta = \pi \]

\[ 0 = \frac{1}{b} \sin (\pi - \theta) + \frac{D}{2b^2} \left[ \cos (\pi - \theta) - 1 \right] \]

\[ 0 = \frac{1}{b} (\sin \theta) + \frac{D}{2b^2} (-\cos \theta - 1) \implies b = \frac{D}{2} \frac{(\cos \theta + 1)}{\sin \theta} \]
\[ b = \frac{D}{2} \left( \frac{\cos^2 \left( \frac{\Theta}{2} \right) - \left[ 1 - \cos^2 \left( \frac{\Theta}{2} \right) \right] + 1}{2 \sin \left( \frac{\Theta}{2} \right) \cos \left( \frac{\Theta}{2} \right)} \right) = \frac{D}{2} \cot \left( \frac{\Theta}{2} \right) \]

\[ b = \frac{D}{2} \cot \left( \frac{\Theta}{2} \right) \]

An \( \alpha \)-particle inside the area \((2\pi b) \, db\) will be scattered between the angles \(\Theta\) and \(\Theta + d\Theta\).

\[ \text{\# of nuclei} \left( \frac{2\pi b \, db}{\text{unit area}} \right) = \frac{P(b) \, db}{P(\Theta) \, d\Theta} \]

probability that an \( \alpha \)-particle will be between "\( b \)" and "\( b + db \) of a nucleus

\[ t = \text{thickness of the foil} \]

\[ \frac{\text{\# of nuclei}}{\text{volume}} \Rightarrow \frac{\text{\# of nuclei}}{\text{unit area}} = \text{\# of nuclei} \cdot t \]
\[ P(b) \, db = \int \frac{2 \pi b \, db}{\sin(\theta)} \]

\[ b = \frac{D}{2} \cot\left(\frac{\theta}{2}\right) \Rightarrow db = \frac{D}{2} \left(\frac{-\sin^2\theta - \cos^2\theta}{\sin^2(\theta/2)}\right) \frac{d\theta}{\sin^2(\theta/2)} = \frac{-D \, d\theta}{4 \, \sin^2(\theta/2)} \]

\[ b \, db = \frac{D}{2} \cot\left(\frac{\theta}{2}\right) \left(\frac{-D}{4}\right) \frac{d\theta}{\sin^2(\theta/2)} = -\frac{D^2}{8} \frac{\cos(\theta/2) \sin(\theta/2)}{\sin^3(\theta/2)} \, d\theta \]

\[ b \, db = -\frac{D^2}{16} \frac{\sin \theta \, d\theta}{\sin^3(\theta/2)} \]

\[ P(b) \, db = -\int \frac{2 \pi D^2}{16} \frac{\sin \theta \, d\theta}{\sin^3(\theta/2)} = -\frac{\pi}{8} \int \frac{9 \, \sin \theta \, d\theta}{\sin^3(\theta/2)} \]

\[ \rho(\theta) \, d\theta = \frac{\pi}{8} \frac{\rho \, D^2}{\sin^3(\theta/2)} \, \sin \theta \, d\theta \]

\[ I_o = \frac{\text{# of } \alpha\text{-particles}}{\text{area}} \Rightarrow \rho(\theta) = \frac{N(\theta)}{I_o} \]

\[ \frac{N(\theta) \, d\theta}{I_o} = \frac{\pi}{8} \frac{\rho \, D^2}{\sin^3(\theta/2)} \, \sin \theta \, d\theta \]
\[
\frac{N(\theta) \, d\theta}{I_0} = \frac{N(\Omega) \, d\Omega}{I_0} = \frac{\rho \cdot t \cdot D^2}{16 \sin^4 \left( \frac{\theta}{2} \right)} \times 2\pi \sin \theta \, d\theta
\]

\[
\frac{N(\Omega)}{I_0} = \frac{\rho \cdot t \cdot D^2}{16 \sin^4 \left( \frac{\theta}{2} \right)} = \frac{\rho \cdot t \cdot \left( \frac{q_k \cdot Z \cdot e^2}{m_k \cdot v^2} \right)^2}{16 \sin^4 \left( \frac{\theta}{2} \right)} = \frac{\rho \cdot t \cdot \left( \frac{q_k \cdot Z \cdot e^2}{m_k \cdot v^2} \right)^2}{4 \sin^4 \left( \frac{\theta}{2} \right)}
\]

\[
N(\Omega) = \frac{\rho \cdot t \cdot \left( \frac{q_k \cdot Z \cdot e^2}{2 \cdot E_k} \right)^2}{\sin^4 \left( \frac{\theta}{2} \right)}
\]

as \( q_k = 2e \)

\( m_k \cdot v^2 = 2E_k \)

Rutherford's equation

* Kinetic energy of an \( \alpha \)-particle

* \( N \propto \sin^9 \left( \frac{\theta}{2} \right) \) confirmed over 5 orders of magnitude \((10^5)\)

* Dependence \( N \propto t \) confirmed over a factor 10

* Confirmed with several kinetic energy from different radioactive sources

* Allowed determination of \( Z \) for several atoms
Closest approximation:

\[ \frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{D}{2b^2} \left( \cos \varphi - 1 \right) \]

But, at closest approximation, \( r = r_d \), \( \theta + 2\varphi = \pi \Rightarrow \varphi = \frac{\pi}{2} - \frac{\theta}{2} \)

\[ \frac{1}{r_d} = \frac{1}{b} \cos \left( \frac{\theta}{2} \right) + \frac{D}{2b^2} \left[ \sin \left( \frac{\theta}{2} \right) - 1 \right] \]

And \( b = \frac{D}{2} \cot \left( \frac{\theta}{2} \right) \)

\[ \frac{1}{r_d} = \frac{2}{D} \tan \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) + \frac{D}{2} \left[ \frac{2}{D} \tan \left( \frac{\theta}{2} \right) \right]^2 \left[ \sin \left( \frac{\theta}{2} \right) - 1 \right] \]

\[ \frac{1}{r_d} = \frac{2}{D} \sin \left( \frac{\theta}{2} \right) + \frac{2}{D} \frac{\sin^2 \left( \frac{\theta}{2} \right)}{\left[ 1 - \sin^2 \left( \frac{\theta}{2} \right) \right]} \left[ \sin \left( \frac{\theta}{2} \right) - 1 \right] \]

\[ \frac{1}{r_d} = \frac{2}{D} \left[ \frac{\sin \left( \frac{\theta}{2} \right)}{\left[ 1 - \sin^2 \left( \frac{\theta}{2} \right) \right]} + \sin^2 \left( \frac{\theta}{2} \right) \left[ \sin \left( \frac{\theta}{2} \right) - 1 \right] \right] \]
\[
\frac{1}{r_d} = \frac{2}{D} \left\{ \sin\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \right\} = \frac{2}{D} \frac{\sin\left(\theta/2\right)}{\left[1 - \sin^2\left(\theta/2\right)\right]} \left[ \frac{1 - \sin^2\left(\theta/2\right)}{1 + \sin^2\left(\theta/2\right)} \right]
\]

\[
\frac{r_d}{D} = \frac{D}{2} \left[1 + \frac{1}{\sin\left(\theta/2\right)}\right]
\]

"Head-on" collision: \( \theta = 0 \), then:

\[
\theta = 0 \Rightarrow \sin\left(\frac{\theta}{2}\right) = 1 \Rightarrow \frac{r_d}{D} = 1
\]

\[
r_d = D = 2 \frac{ke^2}{m_0 v^2} \quad \Leftrightarrow \quad \frac{ke^2}{m_0 v^2} = \frac{1}{2} m_0 v^2
\]

At closest approximation of a "head-on" collision, all the initial kinetic energy is converted into (electric) potential energy.
Atomic Spectra:

* Atoms emit and absorb discrete wavelengths (frequencies) of light.

* Unlike blackbody thermal radiation, which is continuous. For black body, \( R(\lambda) \) is a continuous function of \( \lambda \). Energy was indeed discretized, but all wavelengths (frequencies) are still present.

* A discrete set of spectral lines \( \leftrightarrow \) a particular atom (spectrum)

Experimental evidence:

* Balmer series: \( \lambda_n = 364.6 \frac{n^2}{n^2 - 4} \) nm, \( n = 3, 4, 5, \ldots \) (empirical law for hydrogen atom, 1885)

* Rydberg-Kite formula: \( \frac{1}{\lambda} = \frac{1}{R_H} \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \) with \( n > m \) (empirical law for hydrogen atom, 1913)

\( R_H: \) Rydberg constant, \( R_H = 1.096776 \times 10^7 \) m

* Other atoms: \( \frac{1}{\lambda_{m,n}} = R \left[ \frac{1}{(m-a)^2} - \frac{1}{(n-b)^2} \right] \), \( R \equiv R_H \)
* Electron in a circular orbit due to Coulomb force:

\[ \begin{align*}
E &= \frac{KZe^2}{r^2} = \frac{mv^2}{r} \\
E &= \frac{1}{2}mv^2 + \left( -\frac{KZe^2}{r} \right) \\
but, \text{ from (1): } \quad m\,v^2 &= \frac{KZe^2}{r}
\end{align*} \]

\[ E = \frac{1}{2} \frac{KZe^2}{r} - \frac{KZe^2}{r} = -\frac{1}{2} \frac{KZe^2}{r} \]

\[ E = -\frac{KZe^2}{2r} \]

\[ W = 2\pi \nu = \frac{\nu}{r} = \left( \frac{KZe^2}{m\,r} \right)^{1/2} \cdot \frac{1}{r} = \left( \frac{KZe^2}{m} \right)^{1/2} \cdot \frac{1}{r^{3/2}} \]

Classically (electromagnetic theory):

* the accelerated electron will emit electromagnetic radiation, which will take energy away.

* electron orbit will spiral towards the nucleus and collapse!!
Niels Bohr (1913): model of the hydrogen atom

Bohr’s postulates:

1. An electron in an atom moves in a circular orbit about the nucleus under the influence of the Coulomb attraction between the electron and the nucleus, obeying the laws of classical mechanics.

2. Instead of the infinite number of orbits which would be possible in classical mechanics, it is only possible for an electron to move in an orbit for which its orbital angular momentum, \( L \), is an integral multiple of \( \frac{\hbar}{2\pi} \) (Planck’s constant divided by \( 2\pi \), \( \hbar = \frac{h}{2\pi} \)).

\[ L = n \frac{\hbar}{2\pi} \]

3. Despite the fact that it is constantly accelerating, an electron moving in such an allowed orbit does not radiate electromagnetic energy. Thus, its total energy \( E \) remains constant.

4. Electromagnetic radiation is emitted if an electron, initially moving in an allowed orbit of total energy \( E_i \), discontinuously changes its motion so that it moves to another allowed orbit of total energy \( E_f \). The frequency of the emitted (absorbed) radiation is given by:

\[ \nu = \frac{E_i - E_f}{\hbar} \]
From the Coulomb force for a circular motion and postulate (2):

\[ \frac{KZe^2}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{KZe^2}{mr} \Rightarrow \nu = \left( \frac{KZe^2}{mr} \right)^{1/2} \]

\[ L = mr\nu = mr \cdot \left( \frac{KZe^2}{mr} \right)^{1/2} = \nu R \Rightarrow \frac{m r^2 KZe^2}{r} = \nu^2 \frac{\hbar^2}{2m} \]

\[ \frac{1}{r} = \frac{\hbar^2}{mKZe^2} \Rightarrow \frac{1}{\nu} = \frac{\hbar^2}{mKZe^2} = \frac{n^2 a_0}{Z} \quad \text{(discrete orbits)} \]

\[ a_0 = \frac{\hbar^2}{mKZe^2} = 0.0529 \text{ nm} = 0.529 \text{ Å} = 0.529 \times 10^{-10} \text{ m} \quad \text{(Bohr radius)} \]

When \( Z = 1 \) (hydrogen), the smallest \( r_n \) occurs for \( n = 1 \)

\[ r_1 = a_0 \]

Now, by applying the results above to the total energy:

\[ E = -\frac{KZe^2}{2r} = -\frac{KZe^2}{2} \left( \frac{mKZe^2}{n^2 \hbar^2} \right) = -\left( \frac{mK^2Ze^4}{2\hbar^2} \right) \frac{1}{n^2} \Rightarrow \]

\[ E_n = -\frac{Ze^2 E_0}{n^2}, \quad \text{where} \ E_0 = \frac{mK^2e^4}{2\hbar^2} = 13.6 \text{ eV} = 2.79 \times 10^{-18} \text{ J} \]

(discrete energy levels)
Energy Levels for Hydrogen Atom \((\mathcal{Z} = 1)\): \(E_n = \frac{-E_o}{n^2}\)

\[\begin{array}{ll}
n = \infty & E = 0 \\
n = 4 & E = -0.85 \text{ eV} \\
n = 3 & E = -1.51 \text{ eV} \\
n = 2 & E = -3.40 \text{ eV} \\
n = 1 & E = -13.6 \text{ eV} \\
\end{array}\]

Those energies then determine the frequencies (wavelengths) of the emitted electromagnetic radiation according to postulate (4):

\[E_i - E_f = \Delta E = -\mathcal{Z}^2 E_o \left( \frac{1}{\eta_i^2} - \frac{1}{\eta_f^2} \right) = h \nu = \frac{h \, c}{\lambda}\]

\[\left[ \frac{1}{\lambda} = \frac{\mathcal{Z}^2 E_o}{h \, c} \left( \frac{1}{\eta_f^2} - \frac{1}{\eta_i^2} \right) \right]\]

* reproduces Rydberg–Ritz empirical law
Derives Rydberg constant from fundamental constants:

\[ R = \frac{E_r}{\hbar c} = \frac{m k^2 e^q}{2 \pi^2} \frac{1}{2 \pi^2 \hbar c} = \frac{m k^2 e^q}{4 \pi^2 \hbar^3 c} \approx 1.097 \times 10^7 \text{ m} \]

Predicted new spectra series that were unknown at that time.

Balmer series: \( \eta_f = 2 \) \{ already known at the time of Bohr’s theory \}

Paschen series: \( \eta_f = 3 \) \}

Lyman series \( \eta_f = 1 \)

Brackett \( \eta_f = 4 \) \{ discovered after the predictions of Bohr’s theory \}

Pfund \( \eta_f = 5 \) \}
$\text{He}^+$: singly ionized helium atom:

$$\frac{1}{\lambda} = \frac{Z^2 E_0}{hc} \left( \frac{1}{\eta_f^2} - \frac{1}{\eta_i^2} \right)$$

$$2 = 2 \quad \Rightarrow \quad 2^2 = 4 \quad \Rightarrow \quad \frac{1}{\lambda_{\text{He}^+}} = \frac{4}{\lambda_{\text{H}}} \quad \Rightarrow \quad \lambda_{\text{He}^+} = \frac{\lambda_{\text{H}}}{4}$$

**Absorption:**

- only photons with energy $h\nu$ that matches a particular jump between two allowed energy levels can be absorbed by the atom.

$$E_f - E_i = h\nu$$

- as $E_i$ is usually at the ground state ($n=1$), typically only the spectral lines associated with this series will be observed in absorption. Example, typically only the Lyman series is observed in the absorption spectra of H.
Correction for finite nuclear mass:

\[ E_K = \frac{p^2}{2m} + \frac{p_{\text{nuc}}^2}{2M} \]

\[ \text{From conservation of momentum: } p = p_{\text{nuc}} \Rightarrow \]

\[ E_K = \frac{p^2}{2} \left( \frac{1}{m} + \frac{1}{M} \right) \Rightarrow \frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} \Rightarrow E_K = \frac{p^2}{2\mu} \]

\[ m \Rightarrow \mu: \quad R = \frac{\mu k^2 e^4}{4\pi \hbar^3 c} \]

\[ R = \frac{k^2 e^4}{\left( \frac{1}{m} + \frac{1}{M} \right) 4\pi \hbar^3 c} \times \frac{m}{M} = \frac{R_\infty}{1 + \frac{m}{M}} \]

\[ R_\infty = \frac{m k^2 e^4}{4\pi \hbar^3 c} = 1.097 \times 10^7 \frac{1}{m} \]

* After finite mass correction, the agreement between \( R \) and spectroscopic data is about \( 3 : 100,000 \).
Frank and Hertz Experiment: (1914)

- Direct (non-optical) verification of quantized energy levels
- Discrete energy levels in more complex atoms (with multiple electrons)

Quantization Rule:

\[ L = n \hbar \Rightarrow m_{\text{rot}} r = p r = \frac{h r}{\lambda} = \frac{n}{2\pi} \]

\[ 2\pi r = n \lambda = \text{perimeter of circle of radius } \cdot r\cdot \]

A critique of the Old Quantum Theory:

* Theory lacks coherence
* Fails to predict more complex atoms (e.g., He)
* No prediction on strength of different transition lines

\[ \Downarrow \]

Schroedinger Theory of Quantum Mechanics