Volume Holograms

(Kogelnik’s Coupled Wave Theory)

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Coupled wave theory can predict the response of elementary volume holograms to coherent illumination. Coherent light incident at or very near the Bragg angle on a volume (deep) hologram can result in diffraction efficiencies approaching 100%. As opposed to plane holograms which are only capable of efficiencies of less than 40%.

The high efficiencies obtained with thick dielectric holograms is important for microimaging, high-capacity information storage and for the practical use of holographic optical component. This technique can also be the basis for experimental work on photo-activated chemical species.

In this review the basic relations will be derived and results will be presented with respect to sinusoidal modulations of both the index of refraction\(n\) and the adsorption constants\(\alpha\) and \(\alpha_1\).

It is important to note that this theory only holds for first order Bragg diffraction at or very near the Bragg angle.

**Derivation of the Coupled Wave Equation**

Beginning with Maxwell’s equations for a non-magnetic material of permeability \(\mu = 1\) the wave equation that relates \(E\) to the complex amplitude \(a(x, z)\) is given by:

\[
(\Delta)^2 E - \mu_0 \sigma \left( \frac{\partial E}{\partial t} \right) - \mu_0 \varepsilon_0 \varepsilon \left( \frac{\partial E}{\partial t} \right)^2 = 0
\]  

(1)

Where the plane wave is propagating in the \(z\) direction with linear polarization in the \(y\) direction. And \(\sigma\) is the conductivity of the hologram medium, \(\mu_0\) is the permeability of free space, and \(\varepsilon_0\) and \(\varepsilon\) are the dielectric constants of free space and of the medium respectively. The solution to (1) has the form:

\[
E(x, z, t) = \text{Re} \{ a(x, z) \exp(i\omega t) \}
\]

Where \(a(x, z)\) is the complex amplitude.  

(2)

Inserting (2) into (1) and taking the partials with respect to time gives (1) in terms of the complex amplitude \(a\). Which is the amplitude of the \(y\) component of the electric field that oscillates with an angular frequency \(\omega\).

\[
(\Delta)^2 a - i \omega \mu_0 \sigma a - \mu_0 \varepsilon_0 \varepsilon \omega^2 a = 0
\]

(3)

Which is the Helmholtz equation when we let \(k^2 = i \omega \mu_0 \sigma - \mu_0 \varepsilon_0 \varepsilon \omega^2\). \(k\) depends on \(x, z\) in our model and is spatially modulated and related to the relative dielectric constant \(\varepsilon(x, z)\) and the conductivity \(\sigma(x, z)\) by the dispersion relation.
\[ k^2 = i\omega\mu_0\sigma - \mu_0\varepsilon_0\varepsilon_0 = (\omega/c)^2 - i\omega\mu\sigma \] (4)

In our model the fringes of the hologram grating result from the spatial modulations of \( \varepsilon \) or \( \sigma \).

Define the relative dielectric constant to have an average component \( \varepsilon_0 \) and a sinusoidially varying component of amplitude \( \varepsilon_1 \). Additionally, define \( \mathbf{r} = xi + yj + zk \) to be the position vector from the origin to anywhere within the medium. Then the spatial distribution of the dielectric constant are the loci of constant \( \varepsilon \) are planes expressed by:

\[ \mathbf{r} \cdot \mathbf{n} = \text{constant} \] (5)

where \( \mathbf{n} \) is the normal to the planes and is given by \( \mathbf{K}/K \), where \( \mathbf{K} \) is the grating vector, and \( a2\pi(\mathbf{r} \cdot \mathbf{n}/d) = \mathbf{K} \cdot \mathbf{r} \) where \( \mathbf{r} \cdot \mathbf{n} \) is the distance separating the plane of constant \( \varepsilon \) from the origin and \( d \) is the spacing corresponding to an increment of 2\( \pi \) radians of phase. An analogous argument holds for the conductivity. We can define the relative dielectric constant and conductivity as:

\[ \varepsilon = \varepsilon_0 + \varepsilon_1\cos(\mathbf{K} \cdot \mathbf{r}) \] (6)

\[ \sigma = \sigma_0 + \sigma_1\cos(\mathbf{K} \cdot \mathbf{r}) \] (7)

Inserting (6) and (7) into (4) gives \( k^2 \) in terms of the relative dielectric constant and conductivity:

\[ k^2 = \beta^2 - 2\alpha\beta + 2\kappa[\exp(i\mathbf{K} \cdot \mathbf{r}) + \exp(-i\mathbf{K} \cdot \mathbf{r})] \] (8)

Where \( \alpha \), \( \beta \) and \( \kappa \) are given by the following relations:

\[ \alpha = \mu\varepsilon_0/2(\varepsilon_0)^{1/2} ; \quad \beta = 2\pi(\varepsilon_0)^{1/2}/\lambda ; \quad \kappa = \pi[2\pi\varepsilon_1/(\varepsilon_0)^{1/2}/\lambda - i\mu\varepsilon_1/(\varepsilon_0)^{1/2}] \]

And equation (3) can be written as:

\[ (\text{Del})^2 \mathbf{a} + k^2 \mathbf{a} = 0 \] (3*)

Where \( \alpha \) and \( \beta \) are the average absorption and propagation constants respectively and \( \kappa \) is the coupling constant for the reading wave \( \mathbf{R} \) and the signal wave \( \mathbf{S} \). \( \kappa \) is the central parameter in the coupled wave theory and as such when \( \kappa = 0 \) there is no coupling and no diffraction.

Since we are dealing with an optical media it is customary to express the dielectric constant in terms of the index of refraction. It is possible to express this system in these terms with the following restrictions imposed:

\[ \alpha \ll 2\pi n/\lambda ; \quad \alpha_1 \ll 2\pi n_1/\lambda , \quad \alpha_1 = \mu\varepsilon_1/(\varepsilon_0)^{1/2} ; \text{and } n_1 \ll n \]

\[ \kappa = \pi n_1/\lambda - i\alpha_1/2 \] (9)

The spatial modulation of \( \alpha_1 \) or \( n_1 \) forms a grating which couples the two waves together and leads to an energy exchange between them. These waves can be described by their complex
amplitudes $R(z)$ and $S(z)$ which vary along $z$ as a result of this energy interchange or loss from absorption. The total electric field in the grating is a superposition of the two waves given by:

$$a = R(z)e^{i\rho \cdot r} + S(z)e^{i\sigma \cdot r}$$  \hspace{1cm} (10)

Where $\rho$ is the propagation vector of the reading wave and $\sigma$ is the propagation vector of the signal wave.

We are now ready to solve the differential equation (3) with the relevant parameters. In order to solve there are two simplifying assumptions we must make: First: The angle of incidence must be close to those satisfying the Bragg condition, it is in this region that appreciable diffraction occurs. Second: Only two waves propagate in the hologram, $R$ which is the reading wave, and $S$ which is the signal wave.

### Bragg Condition

When light illuminating the hologram is incident at the Bragg angle the relation between $\rho$ and $\sigma$ is given by:

$$\sigma = \rho - K$$  \hspace{1cm} (11)

and the vector relation (11) between incident, diffracted and grating vectors take on particular importance. For Bragg incidence $\rho$ and $\sigma$ each form an angle $\Theta_0$ with the crest planes of the sinusoidal hologram grating. $K$ lies in the $xz$ plane perpendicular to the crest planes. Since the electric field vector of the incident wave is polarized in the $y$ direction, $\rho$ is also constrained to the $xz$ plane. Equation (11) constrains $\sigma$ to also lie in the $xz$ plane. The triangle formed by these three co-planar vectors with $\rho$ and $\sigma$ each making an angle $\Theta_0$ with the scattering planes is isocles, $\rho = \beta = \sigma$.

$$K/2 = \rho \sin(\Theta_0) \hspace{1cm} \text{The Bragg Condition}$$  \hspace{1cm} (12)

### The Wave Equation

Inserting (10) into (3*) and performing the indicated operations gives two coupled second order differential equations in terms of $R$ and $S$. Note that all primes are with respect to $z$:

$$R'' - 2i\rho R' - \rho^2 R + \beta^2 R - 2i\alpha \beta R + 2\kappa \beta S = 0 \hspace{1cm} (13)$$

$$S'' - 2i\sigma S' - \sigma^2 S + \beta^2 S - 2i\alpha \beta S + 2\kappa \beta R = 0 \hspace{1cm} (14)$$

The fast variations in the wave functions (10) are contained in the phase factors. $R$ and $S$ change relatively slowly. If $R$ and $S$ change slowly enough we can neglect the $R''$ and $S''$ terms in (13).
We will show this is true shortly. Note that the third and fourth terms in (13) cancel by the Bragg condition. Before applying the same condition to the third and fourth terms of (14) we will evaluate for slight deviations to the Bragg angle of incidence. Let \( \Theta = \Theta_0 + \Delta \) and apply the Bragg condition when \( \Delta \ll \Theta_0 \).

After expanding and using the relation that the angle between \( \rho \) and \( K = \pi/2 - \Theta \):

\[
\beta^2 - \sigma^2 = 2\beta^2 \Delta \sin(2\Theta_0)
\]

Defining the parameter \( \Gamma = \beta \Delta \sin(2\Theta_0) \) gives \( \beta^2 - \sigma^2 = 2\beta\Gamma \). By neglecting \( R'' \) and \( S'' \) (13) and (14) become

\[\begin{align*}
C_r R' + \alpha R & = -iK S \\
C_r = \rho_i/\beta \\
C_r S' + (\alpha + i\Gamma) S & = -iK R \\
C_s = \sigma_i/\beta
\end{align*}\] (15)

The physical processes that effect diffraction are now evident by the system of coupled equations. For every incremental distance \( dz \) propagated by the reading and signal waves through the medium we see three factors affect the complex amplitudes to change by \( dR \) and \( dS \). Absorption, coupling of one wave to another and in the case of the signal wave a phase difference is introduced when the angle of incidence deviates from the Bragg angle. Accumulation of extra phase will force the diffracted wave out of sync with the incident wave and diffraction will cease.

To solve for \( R(z) \) and \( S(z) \) is straight forward, solve (15) for \( S \) and insert the result into (16)

\[
R'' + (\alpha/C_r + \alpha/C_s + i\Gamma/C_s) R' + (\alpha^2 + i\Gamma \alpha + \kappa^2) R/C_r C_s = 0
\] (17)

Which has a solution of the form \( R(z) = \exp[Yz] \) (18)

Inserting (18) into (17) and taking the derivatives gives a quadratic equation for \( Y \) such that

\[
Y^2 + (\alpha/C_r + \alpha/C_s + i\Gamma/C_s) Y + (\alpha^2 + i\Gamma \alpha + \kappa^2) / C_r C_s = 0
\]

\[ Y_{1,2} = -1/2(\alpha/C_r + \alpha/C_s + i\Gamma/C_s) \pm \frac{1}{2}\sqrt{[\alpha/C_r - \alpha/C_s - i\Gamma/C_s]^2 - 4\kappa^2/C_r C_s}\] (19)

Where \( Y_1 \) corresponds to + radical term and \( Y_2 \) corresponds to – radical term

\[ R(z) = R_1 \exp[Y_1 z] + R_2 \exp[Y_2 z] \text{ and } S(z) = S_1 \exp[Y_1 z] + S_2 \exp[Y_2 z] \] (20)

At this point we can justify neglecting the \( R'' \) in comparison to \( \rho_i R' \) and \( S'' \) in comparison to \( \sigma_i S' \) terms in equations (13) and (14). From (18) we see that \( R'' = Y^2 \exp[Yz] \) and from (15) that \( \rho_i R' = Y\beta \cos(\psi) \exp[Yz] \), where \( \psi \) is the angle \( \rho \) makes with the \( z \) axis. For \( \psi < \pi/2 \), \( R'' \ll \rho_i R' \) implies that \( Y \ll \beta \). From (19) we see that if \( \Gamma \) is small then \( Y \ll \beta \) providing the inequalities for \( \alpha \), \( \alpha_1 \) and \( n_1 \) are satisfied. A same evaluation will hold for \( S'' \).
Results for Transmission Holograms

Consider an illuminating wave from the left incident on the transmission hologram. Let the incident wave be normalized such that its amplitude is unity. At \( z = 0 \) we have that \( R(0) = 1 \). Additionally we will have that \( S(0) = 0 \). From (20) we get:

\[
R(0) = R_1 + R_2 = 1 \quad \text{and} \quad S(0) = S_1 + S_2 = 0
\]

\[
S'(0) = Y_1S_1 + Y_2S_2
\]

Substituting these values into (16) gives

\[
S_1 = -S_2 \quad \text{at} \quad z = 0 = -i\kappa / |C_s(Y_1 - Y_2)|
\]

and

\[
S(d) = i\kappa / |C_s(Y_2 - Y_1)| \left[ \exp[Y_2d] - \exp[Y_1d] \right] \quad (21)
\]

If we only consider the orientation of the grating planes to be perpendicular to the hologram surface, then \( \mathbf{K} \) is parallel to the surface. With these conditions \( C_R = C_S = \cos(\Theta_0) \).

**Phase Grating: \( \alpha_1 = \alpha = 0 \)** Diffraction from spatial modulation of \( n \)

Define two parameters for the analysis

\[
\nu = \kappa d / \cos(\Theta_0) = \pi n_1 d / \lambda \cos(\Theta_0) \quad \text{and} \quad \zeta = \Delta \beta d / (\sin(\Theta_0)) = \Gamma d / (\cos(\Theta_0))
\]

From (21) we get

\[
(Y_2 - Y_1) = (2i/d)(\zeta^2 - \nu^2)^{1/2} \quad \text{and} \quad Y_{1,2}d = -i\zeta \pm i(\zeta^2 - \nu^2)^{1/2}
\]

\[
S(d) = -i[\exp[-i\zeta] \sinh((\zeta^2 - \nu^2)^{1/2})]/(1 + \zeta^2 / \nu^2)^{1/2}
\]

**Absorption Transmission Holograms: \( \varepsilon_1 = 0 \) \( \alpha_1 \) and \( \alpha \) finite

\[
\nu = \nu_a = \alpha_1 d / (2 \cos(\Theta_0)) \quad \text{and} \quad \zeta = \Delta \beta d / (\sin(\Theta_0)) = \Gamma d / (\cos(\Theta_0))
\]

\[
= (Y_1 - Y_2) = 2/d(\nu_a^2 - \zeta^2)^{1/2} \quad \text{and}
\]

\[
Y_{1,2}d = -\alpha_1 d / (\cos(\Theta_0)) = -i\zeta \pm (\nu_a^2 - \zeta^2)^{1/2}
\]

\[
S(d) = -\exp[-\alpha_1 d / (\cos(\Theta_0))] \exp[-i\zeta] \sinh((\nu_a^2 - \zeta^2)^{1/2})/(1 + \zeta^2 / \nu_a^2)^{1/2}
\]

Assuming unit amplitude for the incident wave \( R \) at \( z = 0 \), then the diffraction efficiency is given by:

\[
\eta = |S(d)|^2 / |R(0)|^2 = |S(d)|^2
\]
Derivations

Need Bragg's Law \( \rightarrow 2d \sin \theta = \frac{\lambda}{n_0} \)

Maxwell's equations for non-magnetic material of permeability \( \mu = 1 \) and \( \varepsilon = \frac{N_0}{V} \) volume charge:

1) \( \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \)
2) \( \nabla \times \vec{H} = \varepsilon_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \)
3) \( \nabla \cdot \vec{D} = 0 \) (no volume charges)
4) \( \nabla \cdot \vec{H} = 0 \)

curl 1) and substitute 2) and 3)

\( \rightarrow \) wave equation \( \nabla^2 \vec{E} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \) 5)

Assume propagation in \( x \) direction and linear polarization in the \( y \) direction
Assume a planewave solution of the form
\[ E(x, z, t) = \text{Re}\{\tilde{E}(x, z) e^{i\omega t}\} \]

independent of \( y \) and oscillates at a singular frequency \( \omega \). We need to determine \( \tilde{a} \) (amplitude - complex)

Using complex notation and taking the partials w.r.t. time since \( E(x, z) \) is \( \sigma(x, z) \)

\[ \frac{\partial}{\partial t} = -(i\omega) \mu_0 \sigma \tilde{E}(x, z, t) \quad \frac{\partial^2}{\partial t^2} = -(i\omega)^2 \mu_0 \varepsilon_0 c \tilde{E}(x, z, t) \]

\[ = -(i\omega \mu_0 \varepsilon_0 \tilde{a}(x, z) e^{i\omega t} + \omega^2 \mu_0 \varepsilon_0 \tilde{a}(x, z) e^{i\omega t} \]

\[ \nabla^2 E = \nabla^2 \tilde{E}(x, z) e^{i\omega t} \quad \nabla^2 \tilde{a} + \omega^2 \mu_0 \varepsilon_0 \tilde{a} = i\omega M_0 \sigma \tilde{a} \]

Define relative dielectric constant to have an average component \( \varepsilon_0 \) and a sinusoidally varying component of amplitude \( \varepsilon_1 \).

Define \( \hat{r} = \hat{x} x + \hat{y} y + \hat{z} z \) be any point from the origin within the medium → spatial distribution of \( \varepsilon \) the loci of constant \( \varepsilon \) are planes expressed by

\[ \hat{r} \cdot \hat{n} = \text{constant} \]

where \( \hat{n} = \frac{\hat{K}}{K} \)

\[ \hat{E} = \text{Im}(\hat{r} \cdot \hat{a} / d) = \text{K} \cdot \hat{r}, \hat{a} \]

(spatial phase)
and \( \vec{r} \cdot \vec{n} \) is the distance separating the planes of constant \( \varepsilon \) from the origin and \( \delta \) is the spacing (along the normal) of an increment of \( 2\pi \) radians of phase. An analogous treatment applies to \( \sigma \)

\[
\begin{align*}
\varepsilon &= \varepsilon_0 + \varepsilon_1 \cos(\vec{K} \cdot \vec{r}) \quad \text{(7)} \\
\sigma &= \sigma_0 + \sigma_1 \cos(\vec{K} \cdot \vec{r}) \quad \text{(8)}
\end{align*}
\]

\( \Rightarrow \) \( 7, 8 \) \( \Rightarrow \) \( 6 \)

\[
\begin{align*}
&\left[ \omega^2 \mu_0 \varepsilon_0 (\varepsilon_0 + \varepsilon_1 \cos(\vec{K} \cdot \vec{r})) - i \omega \mu_0 (\sigma_0 + \sigma_1 \cos(\vec{K} \cdot \vec{r})) \right] \vec{a} \quad \text{(9)} \\
&= \omega^2 \mu_0 \varepsilon_0 \varepsilon_0 - i \omega \mu_0 \sigma_0 + \left[ \omega^2 \mu_0 \varepsilon_0 \varepsilon_1 - i \omega \mu_0 \sigma_1 \right] \cos(\vec{K} \cdot \vec{r}) \vec{a}
\end{align*}
\]

\[
\omega^2 = \left( \frac{2\pi c}{\lambda} \right)^2, \quad \mu_0 \varepsilon_0 = \frac{1}{c^2} \quad \Rightarrow \text{1st term} = \left( \frac{2\pi \beta}{\lambda} \right)^2 \varepsilon_0 = 13^2
\]

\[
\beta = \frac{2\pi \varepsilon_0^{1/2}}{\lambda} \\
2^{nd} \text{ term} = \frac{2 \pi c \mu_0 \sigma_0}{\lambda} = i \beta c \mu_0 \sigma_0 = i \frac{\beta c \mu_0 \sigma_0}{\varepsilon_0^{1/2}} = 2i \chi \beta
\]

\[
\chi = \frac{c \mu_0 \sigma_0}{2 \varepsilon_0^{1/2}}
\]

\[
3^{rd} \text{ term} = \beta \cdot \beta \cdot \varepsilon_1 , \quad 4^{th} \text{ term} = i \beta c \mu_0 \sigma_1 = 2i \chi \beta
\]

\[
\chi_1 = \frac{c \mu_0 \sigma_1}{2 \varepsilon_0^{1/2}}
\]

\[
\Rightarrow \quad \text{(9)} = \left[ \beta^2 - 2i \chi \beta + \frac{\beta}{\chi} \left( \frac{2 \pi \varepsilon_1}{\lambda \varepsilon_0^{1/2}} - 2i \chi_1 \right) \cos(\vec{K} \cdot \vec{r}) \right] \vec{a}
\]
\[
\text{cos}(K \cdot r) = \frac{e^{i(K \cdot r)} - e^{i(K \cdot r)}}{2}
\]

\[
K = \left( \frac{2\pi e_i}{\lambda e_0^{1/2}} - 2i \kappa \right) \left[ e^{i(K \cdot r)} - e^{i(K \cdot r)} \right]
\]

\[
\beta = \text{average propagation constant} \quad \text{Appears to be } \quad \kappa = \text{the coupling constant between the incident wave } R \text{ and the transmitted wave } S
\]

\[
\star \text{ For } K = 0 \rightarrow \text{no diffraction} \quad \star
\]

\[
\frac{\delta^2}{\gamma^2} + \lambda^2 = 0 \quad K = \left( 3^2 - 2i \kappa \beta + K \beta \right)
\]

we need to describe the optical properties must have a material description in terms of the index of refraction, not the dielectric const.

\[
\psi(z)
\]

solution of this form in a lossless, homogeneous, lossy dielectric medium is

\[
\frac{\lambda^2}{\nu} \rightarrow \nu = i(e_0 + \mu_0 \kappa^2 - i\mu_0 \omega \sigma_1)^{1/2}
\]

\[
e_1 = \sigma_1 = 0 , \mu = 1
\]

\[
\nu = i(3^2 - 2i \kappa \beta)^{1/2}
\]

\[
\chi = \frac{\kappa^2 e_0^{1/2}}{2 e_0^{1/2}}
\]
Assume $\beta \gg \alpha \rightarrow \gamma = (\beta^2 - 2i\alpha \beta)^{1/2}$

True of most optical media

\[ \gamma \approx i\beta \left[ 1 + \frac{1}{2} \left( -\frac{2i\alpha}{\beta^2} \right) + \ldots \right] \approx i\beta + \alpha \quad \beta \gg \alpha \]

\[ f = A e^{-i\beta x} e^{-i\frac{\alpha^2}{2}} \]

So our $\beta = \text{propagation constant}$

$\alpha = \text{absorption constant}$

$\alpha \ll \beta$

Define $n_0 = \text{average index of refraction}$

$\varepsilon \cdot n = \frac{c}{c_0}$

where $c = \text{speed of light in vacuum}$

$c_0 = \text{speed of light in media}$

\[ n_0 = \left( \frac{\varepsilon_0 \mu_0}{\varepsilon_0 \mu_0} \right)^{1/2} \mu = 1 \rightarrow n_0 = (\varepsilon_0)^{1/2} \]

$\beta = \frac{2\pi n_0}{\lambda} \rightarrow \alpha \ll \frac{2\pi n_0}{\lambda} \quad \varepsilon \cdot \alpha \ll \frac{2\pi n_0}{\lambda}$

Likewise $n$ of the hologram $n^2 = \varepsilon$

\[ n^2 = (n_0 + n_1 \cos(\vec{k} \cdot \vec{r}))^2 = \varepsilon_0 + \varepsilon_1 \cos(\vec{k} \cdot \vec{r}) \]

$n_1 \ll n_0 \quad \Rightarrow \quad n = n_0(1 + \frac{n_1}{n_0} \cos(\vec{k} \cdot \vec{r}))$

Expanding \[ n_1 = \frac{\varepsilon_1}{2n_0} = \frac{\varepsilon_1}{2\varepsilon_0^{1/2}} \]

\[ K = \frac{2\pi}{2\varepsilon_0} \left( \frac{2\pi}{\lambda} \varepsilon_1 - 2i\alpha_1 \right) = \frac{2\pi n_1}{\lambda} - i\frac{\alpha_1}{2} \]

\[ K = \frac{2\pi n_1}{\lambda} - i\frac{\alpha_1}{2} \]
Spatial modulation indicated by \( n \), or \( \alpha \), forms a grating which couples \( R \) and \( S \) and exchanges energy between them. These waves are defined by their complex amplitudes \( R(z) \) and \( S(z) \). The solution is a superposition of \( R \) and \( S \),

\[
R(z) e^{-i\hat{\sigma} \cdot \hat{x}} + S(z) e^{-i\hat{\sigma}_0 \cdot \hat{x}}
\]

where \( \hat{\sigma} \) is the propagation vector of the reading wave \( \tilde{\sigma} \) is propagation vector of the signal wave.

\[
\hat{\sigma} = \begin{bmatrix} P_x \\ 0 \\ P_z \end{bmatrix} = \beta \begin{bmatrix} \sin \theta \\ 0 \\ \cos \theta \end{bmatrix}
\]

and \( \hat{\sigma} = \hat{P} - \hat{K} = \begin{bmatrix} 0 \\ 0 \\ \sigma_z \end{bmatrix} = \beta \begin{bmatrix} \sin \theta - \frac{K}{\beta} \sin \phi \\ 0 \\ \cos \theta - \frac{K}{\beta} \cos \phi \end{bmatrix} \)

For the Bragg condition

\( \hat{\sigma} = \hat{\sigma}_0 = \hat{\beta} \)
\[
\text{\textcircled{1}} \quad \sin \theta_0 = \frac{K}{2 \rho} \quad \rho = 2 \beta \\
\sin \Theta_0 = \frac{K}{2 \beta} \\
\rho \sin \Theta_0 = \frac{K}{2} \\
\frac{\rho \sin \Theta_0}{d} \rightarrow 2 \frac{\rho \lambda_0 \sin \Theta_0}{\lambda} = \frac{d}{\lambda} \\
2d \sin \Theta_0 = \frac{1}{\lambda} \\
\hat{a} = \hat{a}_R + \hat{a}_S = \hat{R}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\rightarrow \nabla^2 \hat{a} + \kappa^2 \hat{a} = 0 \\
\nabla^2 \hat{a} = \frac{2}{\partial x} \left[ \frac{\partial \hat{a}}{\partial x} \right] + \frac{2}{\partial y} \left[ \frac{\partial \hat{a}}{\partial y} \right] + \frac{2}{\partial z} \left[ \frac{\partial \hat{a}}{\partial z} \right] \\
\frac{\partial \hat{a}}{\partial x} = \hat{R}(z) (-i \rho_x) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) (-i \sigma_x) e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\frac{\partial^2 \hat{a}}{\partial x^2} = \frac{2}{\partial x} \left[ \frac{\partial \hat{a}}{\partial x} \right] = \hat{R}(z) (-i \rho_x) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) (-i \sigma_x)^2 e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\frac{\partial \hat{a}}{\partial z} = \hat{R}'(z) e^{-i(\hat{\sigma} \cdot \hat{x})} \hat{R}(z) (-i \rho_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\frac{\partial^2 \hat{a}}{\partial z^2} = \hat{R}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}'(z) (-i \rho_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}(z) (-i \rho_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}(z) (-i \rho_z)^2 e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\frac{\partial^2 \hat{a}}{\partial z^2} = \hat{R}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}'(z) (-i \rho_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}(z) (-i \rho_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{R}(z) (-i \rho_z)^2 e^{-i(\hat{\sigma} \cdot \hat{x})} \\
+ \hat{S}(z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) (-i \sigma_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) (-i \sigma_z) e^{-i(\hat{\sigma} \cdot \hat{x})} + \hat{S}(z) (-i \sigma_z)^2 e^{-i(\hat{\sigma} \cdot \hat{x})} \\
= \left[ \hat{R}(z) - 2 i \rho_z \hat{R}'(z) + \hat{R}'(z)^2 \hat{R}(z) \right] e^{-i(\hat{\sigma} \cdot \hat{x})} + \left[ \hat{R}(z) - 2 i \rho_z \hat{S}(z) + i \sigma_z \hat{S}(z) \right] e^{-i(\hat{\sigma} \cdot \hat{x})} \\
\]
\[ K^2 \hat{\alpha} = \beta^2 R(z) e^{-i(\vec{\beta} \cdot \vec{x})} + \beta^2 S(z) e^{-i(\vec{\sigma} \cdot \vec{x})} - 2 i \mu \beta R(z) e^{-i(\vec{\sigma} \cdot \vec{x})} - 2 i \mu \beta S(z) e^{-i(\vec{\sigma} \cdot \vec{x})} \]

\[ + 2 K \beta \hat{\alpha} = 2 K \beta \left[ e^{-i(\vec{K} \cdot \vec{x})} - i(\vec{K} \cdot \vec{\beta}) \right] \left( R(z) e^{-i(\vec{\beta} \cdot \vec{x})} + S(z) e^{-i(\vec{\sigma} \cdot \vec{x})} \right) \]

\[ = 2 K \beta \left[ R(z) e^{-i(\vec{K} \cdot \vec{x})} + S(z) e^{-i(\vec{\sigma} \cdot \vec{x})} \right] \]

These two waves do not satisfy the Bragg condition. \( \vec{\sigma} = \vec{\beta} - \vec{K} \)

We do not consider them further.

\[ R'' - 2 i \sigma_z R' + \sigma^2 R + \beta^2 R - 2 i \alpha \beta R + 2 K \beta S = 0 \quad \rightarrow \quad \rho = \beta \]

\[ S'' - 2 i \sigma_z S' - \sigma^2 S + \beta^2 S - 2 i \alpha \beta S + 2 K \beta R = 0 \quad \text{w.r.t.} \quad z \]

\[ \rightarrow \quad R'' - 2 i \sigma_z R' + 2 K \beta S = 0 \quad \text{(13)} \]

\[ \rightarrow \quad S'' - 2 i \sigma_z S' - 2 i \alpha \beta S + (\beta^2 - \sigma^2) S + 2 K \beta R = 0 \quad \text{(14)} \]

All the fast variations in the wave functions are contained in the phase factors. Furthermore \( \hat{R}(z) \) and \( \hat{S}(z) \) change relatively slowly. Assume the change in \( \hat{R}(z) \) and \( \hat{S}(z) \) are sufficiently slow such that \( R'' \) and \( S'' \) can be neglected. We will show this is valid when we derive \( R(z) \) and \( S(z) \).
For every increment of distance $d\xi$ $R$ and $S$ travel through the thickness of the hologram, the complex amplitude changes by an amount $dR$ or $dS$.

To account for deviation from the Bragg angle $\theta_0$ consider the term $(\beta^2 - \sigma^2)$ in 14. We assume the deviation is small s.t.

$$\Theta = \Theta_0 + \delta \rightarrow (\beta^2 - \sigma^2) = \beta^2 - (\beta - \hat{K})^2$$ (Bragg condition)

$$= \beta^2 - \beta^2 + 2\beta \cdot \hat{K} - \hat{K}^2$$

$$= \beta^2 - \beta^2 = 2\rho K \cos(\frac{\pi}{2} - \Theta) - K^2$$

$$= (\beta^2 - \sigma^2) = 2\rho KS \sin \Theta - K^2$$

$$\sin \Theta = \sin(\Theta_0 + \delta) = \sin \Theta_0 \cos \delta + \sin \delta \cos \Theta_0$$

$\delta$ is small $\Rightarrow$ $\sin \Theta_0 + \delta \cos \Theta_0 \rightarrow \sin \delta \approx \delta$

From $\frac{K}{\alpha} = \rho \sin \Theta_0$ and $\beta = \rho \rightarrow \sin \Theta_0 = \frac{K}{2\beta}$

$$\beta^2 - \sigma^2 \approx 2\rho K \left[ \frac{K}{2\beta} \right] + K^2$$

$$\approx 2\rho K\sigma \cos \Theta_0$$

$$\approx 2\rho \beta (2\cos \Theta_0 \sin \Theta_0)$$

$$\beta^2 - \sigma^2 \approx 2\beta^2 \delta \sin 2\Theta_0$$

Let $\Gamma = \beta \delta \sin 2\Theta_0 \rightarrow \beta^2 - \sigma^2 = 2\beta \Gamma$
Ignoring $R$ and $S$ and letting $(B^2 - \sigma^2) = 2\beta I$

13) \[ -\hat{\alpha} \sigma^2 \hat{R} - \hat{\alpha} i \beta \hat{S} + \hat{\alpha} K \beta \hat{S} = 0 \]
\[ \sigma^2 \hat{R}' + \alpha \beta \hat{R} = -iK \beta \hat{S} \]

14) \[ -\hat{\alpha} \sigma^2 \hat{S}' - \hat{\alpha} i \beta \hat{S} + \hat{\alpha} \beta I \hat{S} + \hat{\alpha} K \beta \hat{R} = 0 \]
\[ \sigma^2 \hat{S}' + (\alpha + i I) \hat{S} = -iK \beta \hat{R} \]

Let \[ \frac{\sigma^2}{\beta} = C_R \] and \[ \frac{\sigma^2}{\alpha} = C_S \]

\[
\begin{align*}
C_R \hat{R}' + \alpha \hat{R} &= -iK \hat{S} \quad \text{and} \quad C_S \hat{S}' + (\alpha + i I) \hat{S} = -iK \hat{R}
\end{align*}
\]

From the final form we see where the changes in the complex amplitude come from:

1) Absorption, \( \alpha \hat{R} \quad \text{and} \quad \alpha \hat{S} \), 2) coupling of \( \hat{R} \leftrightarrow \hat{S} \) (\( K \hat{R}, K \hat{S} \)),
3) \( iI \hat{S} \) term gives an additional phase factor in the diffracted wave. s.t. if \( I \) gets large (deviation from the Bragg angle) then \( \hat{S} \) will be forced out of sync with \( \hat{R} \) and the interaction will cease.

13) and 14) are two first order linear D.E.s

solving 13) for \( \hat{S} \) by substituting in 14)
\[
\frac{c_R \dot{R} + \alpha \dot{R}}{-icR} = S \rightarrow S = \frac{c_R \ddot{R} + \alpha \dot{R}}{-icR}
\]

\[
c_s \left( \frac{c_R \dddot{R} + \alpha \ddot{R}}{-icR} \right) + (\alpha + i\Gamma)(\frac{c_R \dot{R} + \alpha \dot{R}}{-icR}) = -icKR
\]

\[
c_sc_R \dddot{R} + c_s \alpha \ddot{R} + \alpha c_R \dot{R} + \alpha^2 \dot{R} + i\Gamma c_R \dot{R} + i\Gamma \alpha \ddot{R} + K^2 \dot{R} =
\]

\[
= c_s c_R \dddot{R} + \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right) c_sc_R \dot{R} + \left( \alpha^2 + i\Gamma \alpha + K^2 \right) \dot{R} = 0
\]

\[
\rightarrow \dddot{R} + \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right) \ddot{R} + \left( \frac{\alpha^2 + i\Gamma \alpha + K^2}{c_R c_s} \right) \dot{R} = 0 \quad (15)
\]

**Solution 15**

\[
\ddot{R}(z) = e^{\gamma z} \quad \Rightarrow \quad \ddot{R} = \gamma^2 e^{\gamma z} \quad \rightarrow \quad 15)
\]

\[
\ddot{R} = \gamma e^{\gamma z} \quad \dddot{R} = \gamma^2 e^{\gamma z}
\]

\[
\gamma^2 + \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right) \gamma + \left( \frac{\alpha^2 + i\Gamma \alpha + K^2}{c_R c_s} \right) = 0 \quad (16)
\]

**Solving**

\[
\gamma \rightarrow - \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right) \pm \sqrt{\left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right)^2 - 4 \left( \frac{\alpha^2 + i\Gamma \alpha + K^2}{c_R c_s} \right)}
\]

\[
\gamma_{1,2} = - \frac{1}{2} \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right) \pm \frac{1}{2} \sqrt{\left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i\Gamma}{c_s} \right)^2 - 4 \left( \frac{\alpha^2 + i\Gamma \alpha + K^2}{c_R c_s} \right)}
\]

\[
\gamma_1 = + \left[ \ldots \right]^{1/2}, \quad \gamma_2 = - \left[ \ldots \right]^{1/2}
\]

A similar calculation yields the solution for \( \ddot{S}(z) \).
\[ \hat{R}(z) = R_1 e^{\gamma_1 z} + R_2 e^{\gamma_2 z}, \quad \hat{S}(z) = S_1 e^{\gamma_1 z} + S_2 e^{\gamma_2 z} \]

From \( \hat{R}(z) = e^{\gamma_1 z} \) and the expression for \( \gamma_1, \gamma_2 \)

\[ \hat{R} = \gamma_1 e^{\gamma_1 z} \quad \hat{R} = \gamma_2 e^{\gamma_2 z} \quad \hat{R} = \gamma_1 \cos \Theta e^{\gamma_1 z} \quad \hat{R} = \gamma_2 \cos \Theta e^{\gamma_2 z} \]

If \( \Theta < 90^\circ \), then \( \hat{R} \ll \hat{R} \) implies \( \gamma_1 \ll \gamma_2 \)

which is true when \( I \) is very small

\[ I \ll \sqrt{3} \quad \text{and} \quad \sqrt{3} \gg \alpha, \sqrt{3} \gg \alpha', \text{and} \quad n \gg \ell \]

Similar results hold for \( \hat{S} \) w.r.t. \( \hat{S} \)

---

**Results:** Transmission Hologram (phase grating \( \perp \) to \( z = 0 \) \& \( z = T \) surface)

\[
\hat{R}(z) \text{ has unit amplitude s.t. } \\
\hat{R}(0) = \hat{R}_1 + \hat{R}_2 = 1 \\
\hat{S}(0) = \hat{S}_1 + \hat{S}_2 = 0 \Rightarrow \hat{S}_1 = -\hat{S}_2 \\
\hat{S}(0) = \gamma_1 \hat{S}_1 + \gamma_2 \hat{S}_2 \\
\hat{S}(0) = s_1 (\gamma_1 - \gamma_2) c_5 = -iK \\
\hat{R}(0) = 1 \Rightarrow s_1 = \frac{-iK}{c_5 (\gamma_1 - \gamma_2)} \\
\hat{S}(T) = \frac{iK}{c_5 (\gamma_1 - \gamma_2)} \left[ e^{\gamma_2 T} - e^{\gamma_1 T} \right] 
\]
Need $\gamma_1 - \gamma_2$, $\gamma_2 T \leq \gamma_1 T$ @ the Bragg angle

$\rightarrow c_R = \frac{\beta}{\gamma} = c_s = \frac{\alpha}{\beta} = \cos \theta_0$

Need amplitude $\tilde{S}(T)$ for 2 cases

1st case $\alpha = \alpha_1 = 0$ (lossless dielectric)

Define $\tilde{\xi} = T \beta T \sin \theta_0 = \frac{FT}{2 \cos \theta_0}$

$\sqrt{\frac{K^T}{\cos \theta_0}} = \frac{\pi n_i T}{\lambda \cos \theta_0}$

$\gamma_1 - \gamma_2 (\alpha = \alpha_1 = 0) = -\frac{1}{2} \left( \frac{\gamma T}{c_s} \right) + \frac{1}{2} \left[ \left( \frac{i \Gamma}{c_s} \right)^2 - 4 \left( \frac{K^2}{c_R c_s} \right) \right]^{1/2}$

$- \left[ \left( -\frac{1}{2} \left( \frac{\gamma T}{c_s} \right) - \frac{1}{2} \left[ \left( \frac{i \Gamma}{c_s} \right)^2 - 4 \left( \frac{K^2}{c_R c_s} \right) \right] \right]^{1/2}$

$\gamma_1 T = \left[ \left( \frac{i \Gamma}{c_s} \right)^2 - 4 \left( \frac{\pi n_i}{\lambda \cos \theta_0} \right)^2 \right]^{1/2} = \frac{2i}{T} \left[ \theta \tilde{\xi}^2 + \sqrt{-\xi^2} \right]^{1/2}$

$\gamma_2 T = -\frac{1}{2} \left( \frac{i \Gamma}{c_s} \right) T + \frac{1}{2} \left[ \left( \frac{i \Gamma}{c_s} \right)^2 - 4 \left( \frac{\pi n_i}{\lambda \cos \theta_0} \right)^2 \right]^{1/2} = -i \tilde{\xi} \pm (\tilde{\xi}^2 + \nu^2)^{1/2}$

$\tilde{S}(T) = \frac{i \left( \frac{\pi n_i}{\lambda} \right)}{c_s \left( \frac{2i}{T} \right) (\tilde{\xi}^2 + \nu^2)^{1/2}} \left[ e^{-i \tilde{\xi} - i(\tilde{\xi}^2 + \nu^2)^{1/2}} - i \tilde{\xi} + i(\tilde{\xi}^2 + \nu^2)^{1/2} \right]$
\[ \mathbf{\hat{s}}(T) = \frac{\hat{c}}{2i} \left( \frac{\pi n_1 T}{\lambda c_s} \right) \times \frac{\sqrt{1 + \frac{\xi^2}{v^2}}}{\sqrt{1 + \frac{\xi^2}{v^2}}} \]

\[ e^{-i \xi} \left[ e^{-i(\xi + v^2)^{1/2}} - e^{i(\xi + v^2)^{1/2}} \right] = -i \xi \sin(\xi + v^2)^{1/2} \]

\[ \mathbf{\hat{s}}(T) = -i \frac{e^{-i \xi}}{\sin(\xi + v^2)^{1/2}} \frac{\sin(\xi + v^2)^{1/2}}{(1 + \xi^2/v^2)^{1/2}} \]

\[ \mathbf{\hat{\eta}} \text{ (efficiency)} = \frac{\mathbf{\hat{s}}(T)}{\mathbf{\hat{R}}(0)} = \frac{\mathbf{\hat{s}}(T)^2}{\mathbf{\hat{R}}(0)} = 1 \]

Let \( \Theta = \Theta_0 \rightarrow \xi = 0 \times e^i \xi = 0 \)

\[ \mathbf{s}(T) = -i \sin(\xi) \]

We get 100% \( \mathbf{\eta} \) when \( \sin v^2 = 1 \)

or \( v = \frac{\pi n_1 T}{\lambda \cos \Theta_0} = \frac{T}{2} \rightarrow \) Bragg incidence

Only \( n_1 \) determines efficiency \( \rightarrow \propto = 0 \)

\[ \mathbf{n}_1 \frac{T}{\cos \Theta_0} = \frac{\lambda}{2} \rightarrow \text{see graphs for different values of} \]

\[ \star \]

100% when \( T \) large \& \( \Theta \neq \Theta_0 \)

For \( \frac{1}{2} \) wavelength in air

Absorption Transmission \( \rightarrow e_1 = 0 \times \propto \neq 0 \), Finite

\[ \Gamma = -i \chi_1 \frac{1}{2} \]

Now \( v_a = \frac{\chi_1 T}{2 \cos \Theta_0} \), \( \xi = \frac{\pi T}{2 \cos \Theta_0} \)

\[ \chi_1 - \chi_2 = \left[ -\left( \frac{\pi}{c_s} \right)^2 + \frac{\chi_1^2}{c_s c_R} \right]^{1/2} = \left[ \left( \frac{\chi_1}{c_0} \right)^2 - \left( \frac{\pi}{c_0} \right)^2 \right]^{1/2} \]

\[ = \frac{2}{T} \left[ v_a^2 - \xi^2 \right] \]

As before the first terms in \( \gamma_1 \), subtract out
\[ \gamma_{12}^{T} = -\frac{1}{2} \left( \frac{\alpha}{c_R} + \frac{\alpha}{c_s} + \frac{i \Gamma}{c_s} \right) \pm \left[ \left( \frac{\alpha}{c_R} \cos \Theta \pm \frac{i \Gamma}{c_s} \cos \Theta \right)^2 - 4 \left( \frac{\alpha^2}{\cos^2 \Theta} \right) \frac{1}{c_R c_s} \right]^{1/2} \]

\[ = -\frac{1}{2} \left( \frac{2 \alpha}{\cos \Theta} + \frac{i \Gamma}{\cos \Theta} \right) \pm \left( \frac{2 \alpha}{\cos \Theta} + \frac{i \Gamma}{\cos \Theta} \right)^2 - \chi \left( \frac{\alpha^2}{\cos^2 \Theta} \right) \frac{1}{c_R c_s} \right]^{1/2} \]

\[ = -\frac{\alpha T}{\cos \Theta} - \frac{i \Gamma T}{2 \cos \Theta} + \frac{\alpha^2}{\cos \Theta} \cos \Theta \sqrt{\frac{\alpha^2}{\cos^2 \Theta} - \frac{i \Gamma^2 T^2}{\cos^2 \Theta}} \]

\[ = -\frac{\alpha T}{\cos \Theta} - i \xi + \left( \sqrt{\alpha^2 - \xi^2} \right) \frac{T}{\cos \Theta} \]

\[ \gamma_{12}^{T} = -\frac{\alpha T}{\cos \Theta} - i \xi + \left( \sqrt{\alpha^2 - \xi^2} \right) \frac{T}{\cos \Theta} \]

\[ S(T) = -\exp \left[ -\frac{\alpha T}{\cos \Theta} \right] \exp \left[ -i \xi \right] \frac{\sinh \left( \sqrt{\alpha^2 - \xi^2} \right) \frac{T}{\cos \Theta}}{\left( 1 - \frac{\xi^2}{\alpha^2} \right)^{1/2}} \]

\[ \xi = 0 \quad S(T) = -\exp \left[ -\frac{\alpha T}{\cos \Theta} \right] \sinh \left( \sqrt{\alpha^2 - \xi^2} \right) \frac{T}{\cos \Theta} \]

(Bragg incidence) \quad -\exp \left[ -\frac{\alpha T}{\cos \Theta} \right] \sinh \left( \alpha T \right) \frac{1}{\cos \Theta} \]

Similar results are obtained when

\[ \rightarrow \text{Different Boundary Conditions for Reflection Holograms} \]