DENSITY STUDIES OF MHD INTERSTELLAR TURBULENCE: STATISTICAL MOMENTS, CORRELATIONS AND BISPECTRUM

By

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B.S. Physics

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ABSTRACT

DENSITY STUDIES OF MHD INTERSTELLAR TURBULENCE: STATISTICAL MOMENTS, CORRELATIONS AND BISPECTRUM

Blakesley Burkhart

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We performed three-dimensional compressible MHD simulations of interstellar gas for different sonic and Alfvénic Mach numbers. We focused our analysis on the resulting density and column density maps, the latter being available through observations. We calculated the 3rd and 4th statistical moments of these quantities, their correlations with the kinetic and magnetic energies, and obtained the bispectrum in order to characterize the distributions. We used the bispectrum technique to investigate the role of non-linear wave-wave interactions in the turbulent energy cascade. Although the bispectrum has been extensively used in cosmological studies, it has never been examined for the case of astrophysical MHD turbulence. We confirm the high Gaussianity of subsonic models of density and column density, as well as a strong dependence of the skewness and kurtosis with the sonic Mach number ($M_s$). Our results show a correlation of density and column density with magnetic field. This trend is independent of the turbulent kinetic energy and can be used to characterize inhomogeneities of physical properties in low density and clumps in the ISM. We also obtained the bispectrum function for density and column densities with varying magnetic field strength. As expected, strong correlation is obtained for wave modes $k_1 = k_2$ for all models. Larger values
of $M_s$ result in increased correlations for modes with $k_1 \neq k_2$. This effect is even more evident for increasing magnetic field intensity. We believe that the different MHD wave modes, e.g. Alfvén and magneto-acoustic, arising in strongly magnetized turbulence may be responsible for the increased correlations, when compared to purely hydrodynamical perturbations.
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CHAPTER I

INTRODUCTION

"When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first."

Werner Heisenberg

The interstellar medium (ISM) is the matter between the stars consisting of ionized and neutral gaseous hydrogen, helium, and metals and dust. The ISM is extremely diffuse and collisions between atoms can often take up to seconds, which is a very long time on the atomic scale. Interstellar dust causes the extinction of light from distant stars and nebulae which can be a source of problems for astronomers who study stellar objects. However, the ISM is worth study in itself as it is the birth place of stars. In order to understand star formation, observational and theoretical studies of the ISM are critical. What we now see as a star was once part of large gas cloud in the ISM. The star formed as the gas clouds contracted under the influence of gravity. In the absence of any other forces, a spherical gas cloud of radius r and mass M changes its radius according to Newton’s second law:

\[ \ddot{r} = -\frac{GM}{r^2} \tag{1} \]

In equilibrium, a self-gravitating gas cloud is supported against collapse by internal pressure that is due to both thermal and turbulent pressures. However, equilibrium will not be reached until the gravitational free fall time given by: \( t \approx (G\rho)^{-\frac{1}{2}} \) where \( \rho \) is the mean density of the interstellar cloud, has passed. In general a typical cool interstellar cloud has a free-fall time of only a few million years, which is much
shorter than the lifetime of the Milky Way. Since we still observe an abundance of interstellar gas clouds, there seems to be a flaw in this theory. The ISM is very complex, and there are other forces then gravity at play which greatly increase the free fall time. Turbulent pressure and magnetic pressure are two such forces. Studying the interplay between magnetic and gas pressure is crucial to understanding how the structure of interstellar gas evolves over time.

The theory of turbulence, while still considered one of the greatest unsolved problems in physics, has made considerable leaps in the past fifty years. The flow of a gas can be classified as laminar or turbulent depending on its Reynold’s number, which is defined to be the ratio of the inertial forces to viscous forces. A flow with R > 4000 will be turbulent. Turbulence causes formation of eddies in which kinetic energy cascades from large scales to smaller scales in a process that produces a hierarchy of eddies. In a brilliant and original work done in 1941, Andrey Kolmogorov proposed the first static theory of turbulence based on energy cascades. He postulated that for high Reynolds number, small scale turbulence is isotropic (meaning no direction can be determined) and that the scale is determined by the viscosity and rate of energy loss in the fluid. He found that the smallest scales in the turbulent fluid are given by the Kolmogorov length:

\[ L = \left( \frac{\nu^3}{\epsilon} \right)^{\frac{1}{4}} \]  

Where \( \nu \) is the kinematic viscosity of the fluid and \( \epsilon \) is the average rate of energy dissipation per unit mass. This theory is just considers the mean length and time it takes for the energy dissipation. Turbulence is a spatio-temporal phenomenon and thus the energy dissipation changes in space and time.

The ISM is known to be highly turbulent but the role of magnetic fields in the distribution of turbulent eddies, as well as in the energy cascade, is still not completely understood. Numerical simulations have been shown to be important tools to study turbulent processes in the magnetized ISM. Stars are known to be formed on the denser regions of the ISM. However, the formation and survival of
these dense cores are not completely understood, mostly because of the complex relationship between turbulence, magnetic fields and self-gravity [Evans (1999), Elmegreen & Scalo (2004), Scalo & Elmegreen (2004)].

In a rather simplistic view of interstellar turbulence, highly supersonic turbulent motions generate shocks that evolve into dense structures. Subsonic motions are considered incompressible and no shock wave is produced. Magnetic fields may play a key role in the star formation process (see [McKee & Tan (2002), Mac Low & Klessen (2004)]), e.g. providing clouds with an extra support against gravitational collapse [Ostriker et al. (1999)]. Strongly magnetized turbulence (Sub-Alfvénic) present weaker shocks and decreased density contrast compared to purely hydrodynamic (i.e. no magnetic field) or super-Alfvénic (weak magnetic field) turbulence. Because of the difficulties mentioned above, numerical simulations represent a unique method to understand the evolution of turbulence and the role of the magnetic field on the creation of density structures in the ISM.

Recently, [Kowal, Lazarian & Beresnyak (2007)] presented an extensive analysis of density and column density statistics for several different physical conditions of simulated magnetized molecular clouds ($128^3$ and $256^3$ resolutions), involving the comparison of statistical moments and spectra. They also provided a detailed topological analysis using the genus technique. They showed that density and column density PDF’s strongly depend on the local sound speed and on the turbulence regime, in agreement with earlier findings of [Passot & Vázquez-Semadeni (2003)]. Also, they showed that the magnetic field plays an important role in the topology of the dense structures as strongly magnetized clouds present noticeable anisotropies. In this work we present a extension of [Kowal, Lazarian & Beresnyak (2007)] using $512^2$ and $512^3$ resolutions, focusing more on statistical descriptions of turbulence. Despite the apparent success of reproducing the observed distributions, many previous works lack of a detailed
analysis of the statistics of turbulence. Statistical descriptions of turbulence are valuable as they constrain the physical properties of the system. Several techniques involving numerics have been proposed for velocity studies (see [Lazarian, Pogosyan(2004), Lazarian, Pogosyan(2000)]) and have been successfully tested.

Typically, studies of ISM turbulence rely on spectral analysis of waves. Usually, the energy cascade process and the distribution of density structures are analyzed from power spectra, of two-point correlations. On the other hand, the bispectrum or three-point statistic, measures the magnitude and the phase of the correlations of signals in Fourier space. It can remove constraints that two point statistics utilize and provide not only intensity as in spectra but information regarding phases. As a consequence, it can be used to search for nonlinear wave-wave interactions and characterize the turbulent regime. The bispectrum has been widely used in cosmology and gravitational wave studies [Fry(1998), Scoccimarro(2000), Liguori et al.(2006)] and on the characterization of wave-wave interactions on laboratory plasmas [Intrator et al.(1989), Tynan et al.(2001)], but has rarely been applied to ISM MHD turbulence. Not only can the bispectrum be applied to theoretical data, but it can also be used in observational studies to characterize molecular clouds in the ISM.

In this work we will look at several statistics in order to better characterize turbulence in the ISM. It is our goal to be able to relate synthetic data used in studies such as this one to observed data. Techniques studied here could ultimately be useful in characterizing interstellar flows and star forming processes.
CHAPTER II

WHAT CAN BE LEARNED FROM NUMERICAL SIMULATIONS

One may wonder why a theoretical work such as this is important to studies of the interstellar medium. For many years MHD turbulence was a very shaky field in which many models could be presented without means to test them. A study of incompressible MHD turbulence (no shock waves) by Goldreich & Sridhar (1995) was a milestone for the field in that for the first time ever, theoretical predictions were tested with direct 3D numerical simulations. Although observations provide the ultimate testing, numerical simulations provide good resolution testing while a solid theoretical framework provides generalizations for complex phenomena such as astrophysical turbulence. Theoreticians use ideal physical equations to generate data. They then apply various mathematical techniques to analyze the data. While magnetized turbulence is an extremely complex phenomenon, many advances in understanding astrophysical phenomena can be obtained if rather simple statistical measures are known. Statistical measure used in this work include skewness, kurtosis, correlations and bispectrum. Statistical moments have been shown in the past by [Kowal, Lazarian & Beresnyak(2007)] to have dependence on the sonic Mach number, $M_s$. We would like to verify this result, as it can easily be applied by observes to characterize the Mach number of interstellar gas for a given skewness seen in a data set. We also made several correlation plots to see how densities and column densities relate to magnetic energy and specific kinetic energy. These correlations can tell us if any functionality exists and allow us to compact densities with synthetic 2D density. These comparisons between 2D and 3D statistics will
ultimately be used as a basis for observers, since it is generally the case that observers typically measure column densities. Finally the bispectrum is a relatively new technique for astrophysical MHD turbulence. It is our goal that the bispectrum will be used to characterize the flow of turbulent eddies and may be used to determine Mach number and magnetic field strength in observational data.
CHAPTER III

THE CODE

We used a second-order-accurate hybrid essentially nonoscillatory (ENO) scheme (for review see [Cho & Lazarian(2002)]) similar to that shown in [Kowal, Lazarian & Beresnyak(2007)] to solve the ideal MHD equations in a periodic box,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (3)
\]

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{1}{4\pi} \mathbf{B} \mathbf{B} \right] = \rho \mathbf{f}, \quad (4)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \quad (5)
\]

with the zero-divergence condition stating the nonexistence of magnetic monopoles, \( \nabla \cdot \mathbf{B} = 0 \), and an isothermal equation of state \( p = a^2 \rho \), where \( \rho \) is density, \( \mathbf{v} \) is velocity, \( \mathbf{B} \) is magnetic field, \( p \) is the gas pressure, and \( a \) is the isothermal speed of sound. On the right-hand side, the term \( \mathbf{f} \) is a random large-scale driving acceleration. The rms velocity \( \delta \mathbf{v} \) is maintained to be approximately unity, so that \( \mathbf{v} \) is seen as the velocity measured in units of rms velocity of the system and \( \mathbf{B}/(4\pi \rho)^{1/2} \) as the Alfvén velocity in the same units. The time \( t \) is in units of the large eddy turnover time \( (L/\delta \mathbf{v}) \) and the length is in units of \( L \), the size of the box. The magnetic field consists of the uniform background field and a fluctuating field: \( \mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{b} \) where \( \mathbf{b} = 0 \), initially. The magnetic field \( (\mathbf{B}_{\text{ext}}) \) is chosen parallel to the x-direction of the cube.
We ran six compressible MHD turbulent models, with $512^3$ resolution, for $t \sim 5$ crossing times, to guarantee full development of energy cascade. The different initial conditions led to three values of sonic Mach number, $\sim 0.7$, $\sim 2$, and $\sim 7$, and two Alfvenic Mach numbers, $\sim 0.7$ and $\sim 7$. The models are listed and described in Table 1 above.
CHAPTER IV

SKEWNESS AND KURTOSIS

The first and second order moments, known as mean value and variance, are the most widely used in statistical analysis and are given by:

\[ \xi = \frac{1}{N} \sum_{i=1}^{N} \xi_i \]  \hspace{1cm} (6)

\[ \sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\xi_i - \bar{\xi})^2 \]  \hspace{1cm} (7)

where \( N \) is the number of points in the data. Variance of density fluctuations as it is related to \( M_s \) has been well studied (see Nordlunch & Padoan 1990; Ostriker et al. 2001). Highly supersonic turbulence tends to exhibit a broader distribution of density and column density, and the variance may be used to characterize turbulent amplitudes for regions where velocity measurements are scarce. However, these lower order moments are unable to characterize the asymmetry of density and column density distributions, compared to Gaussian. Star formation occurs at the denser part of the PDF, and asymmetries on this distribution may play a role on the star formation efficiency. In this case, we must examine the third and fourth order moments, also known as skewness and kurtosis, respectively. Skewness and kurtosis have been investigated for compressible MHD turbulence in the past (see Kowal, Lazarian & Beresnyak(2007)] henceforth referred to as KLB). In this work we want to check their results by applying these calculations to the current cubes, which have finer resolution (512\(^3\)). The interest in these measurements comes from that fact that skewness of column densities and densities depend strongly on \( M_s \).
Therefore, one can determine $M_s$ from examining the skewness of the observational distributions. Also, by comparing the observed column densities to the synthetic ones and determining the turbulent regime of a given cloud, it is possible to understand the three-dimensional structure. Skewness is defined by the third-order statistical moment as:

$$\gamma_\xi = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\xi_i - \bar{\xi}}{\sigma_\xi} \right)^3$$  \hspace{1cm} (8)

If a distribution is Gaussian, the skewness is zero. Negative skewness indicates the data is skewed in the left direction (the tail is extended to the left) while positive values imply that the distribution is skewed in the positive direction (the tail is extended to the right). Kurtosis is a measure of whether a quantity has a distribution that is peaked or flattened compared to a normal Gaussian distribution. Kurtosis is defined in a similar manner to skewness, only it utilizes the forth order statistical moment. If a data set has positive kurtosis then it will have a distinct sharp peak near the mean and have elongated tails. If a data set has negative kurtosis then it will be flat at the mean. Kurtosis is defined as:

$$\beta_\xi = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\xi_i - \bar{\xi}}{\sigma_\xi} \right)^4 - 3$$  \hspace{1cm} (9)
Table 2 shows the values for the skew and kurtosis of density for models with resolution of $512^3$, as well as the column density maps ($512^2$), with $\mathcal{M}_s$. For a data cube with a given value of $\mathcal{M}_s$, the quantities were averaged over all snapshots. Error bars are determined using the standard deviation of the results for each model. From this data we can see how the skewness of these quantities depend on the Mach number. The skews of density and the column densities are strictly positive. The asymmetry generally grows with increasing $\mathcal{M}_s$ for both column density and density. As expected, small $\mathcal{M}_s$ models are more Gaussian ($\gamma \to 0$), compared to supersonic models. Our results show how the presence of shocks play a role in affecting the asymmetry of densities and column densities. For both density and column densities the asymmetry increases with increasing sonic Mach number. The presence of shocks, which evolve in the high density structures, is enhanced as $\mathcal{M}_s$ increases. The kurtosis of density is shown to be also higher for supersonic models. It is a consequence of the transfer of mass from the average values to the right tail, mostly due to shocks. Again, subsonic models present kurtosis around zero, i.e. similar to Gaussian, in agreement with KLB. The kurtosis of the column densities are strictly positive for $0 \leq \mathcal{M}_s \leq 6.0$. Highly supersonic models are systematically more peaked then subsonic models, following the trend noted from the density distributions.
CHAPTER V

CORRELATIONS

At the most basic level, correlations can be used to determine a possible relationship between two quantities. If two quantities show some sort of functional dependence, one can often predict this behavior for future models and observations. Correlation plots were made for the final snapshots of all models. We made plots of density vs. magnetic energy, density vs. specific kinetic energy, $M_A$ vs. normalized density, and plots of $x$ and $z$ column density vs. magnetic field and $x$ and $z$ column density vs. velocity where $x$ stands for integration parallel to line of sight and $z$ represents integration perpendicular to the line of sight.

The plots of the log density vs. magnetic energy are shown in Figure 1. For supersonic cases we see that the higher the density the higher the magnetic energy, which is in part due to the compressibility of the gas. The compressed regions are dense enough to distort the magnetic field lines, enhance the magnetic field intensity and effectively trap the magnetic energy. The magnetic energy in the super-Alfvénic model also reaches peaks higher than those in sub-Alfvénic cases, due to a larger magnetic pressure in the latter case. For subsonic cases we see a very different trend as a result of incompressible turbulent flows. Due to lack of shocks trapping the magnetic field in density clumps, the magnetic energy is inversely proportional to the density, as expected by Alfvénic perturbations (i.e. $B \propto \sqrt{\rho}$).
Figure 1. The correlation of log density vs specific kinetic energy. The first row consists of super-Alfvenic cases while the bottom row is sub-Alfvenic. Images are ordered from left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of points.
Figure 2. The correlation of log density vs specific kinetic energy. The first row consists of super-Alfvenic cases while the bottom row is sub-Alfvenic. Images are ordered from left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of points.
We show the correlations of log density vs. squared velocity (i.e. the specific kinetic energy) in figure 2. No special relation was obtained. The distributions tend to isotropically distribute around the mean value of density. Therefore, the kinetic energy increases linearly with density. Obviously, subsonic models have minimum kinetic energies at values of log of density that vary less from the mean density than do supersonic models (-0.6 to 0.4 for subsonic and -2 to 2 for supersonic). The subsonic models lack compressibility and reach their minimum kinetic energy with clumps that are less dense than supersonic models.

1 Synthetic correlations for observational comparison

In order to make our studies comparable to the observations, we must provide the correlational studies using column density. We performed this analysis for magnetic field and velocity in order to gain a better understanding of how these quantities interact with turbulent gas. Regarding the velocity field, we focused on correlating the column density to the velocity along the LOS as obtained from spectral lines.

In Figure 3 we show the correlation of column density along x-direction and magnetic field component parallel to the line of sight, with LOS $∥ B_{\text{ext}}$. Similarly to Figure 1, there is an increase in the magnetic field component along the LOS with the column density for the supersonic models. However, sub-Alfvénic models present steeper correlations. Subsonic models showed no correlation.
Figure 3. The correlation of column density in the x direction vs magnetic field parallel to the line of sight. The first row consists of super-Alfvenic models while the bottom row is sub-Alfvenic. Images are ordered left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of points.
Figure 4. The correlation of column density in the $z$ direction vs. magnetic field parallel to the line of sight. The first row consists of super-Alfvénic models while the bottom row is sub-Alfvénic. Images are ordered left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of points.
Figure 4 shows the correlation of column density along z-direction and the magnetic field component parallel to the line of sight. Figure 6 shows the correlation of column density along z-direction and the magnetic field component perpendicular to the line of sight. Compared to Figures 4 and 5, there are similarities for models with \( M_A = 2.0 \), while models with \( M_A = 0.7 \) show less steep correlations. It occurs because, in this case, the line of sight is chosen to be \( \perp B_{\text{ext}} \) and the observed magnetic field is simply the random/perturbed component.

It is also noticeable from Figures 3-6 that supersonic models present smaller dispersion over the column density \( \times \) magnetic field space. In order to compare the observational measurements of velocities from spectral lines, we also studied the correlations of column density and velocity parallel to the line of sight. In Figures 7 and 8 we present correlation of column density along x and z-directions, respectively, to the velocity component parallel to the line of sight.
Figure 5. The correlation of column density in the x direction vs. velocity parallel to the line of sight. The first row consists of super-Alfvenic models while the bottom row is sub-Alfvenic. Images are ordered left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of points.
Figure 6. The correlation of column density in the z direction vs velocity parallel to the line of sight. The first row consists of super-Alfvenic models while the bottom row is sub-Alfvenic. Images are ordered left to right as supersonic to subsonic. Blue contours indicate regions of most frequent points while red and yellow have lower frequency of point.
No correlation is seen for subsonic models in both cases. Actually, considering the range of normalized column density 0 to 1.0, no correlation is seen independently on the model studied. For column densities > 1.0, there is an increase in the dispersion of the velocity projected along the line of sight. Interestingly, this behavior was not observed in figure 2, and may be related to the fact that here we take into account one component of the velocity field, while in figure 2 we computed the specific kinetic energy. This effect is related to the strong anisotropy of velocity regarding the magnetic field lines in low density regions. These regions are confined by the magnetic field and the flows tend to be along it. However, dense clumps can drag the field lines and, therefore, present a more isotropic velocity field. This effect results in an increase in the dispersion of the velocity measured along the line of sight.
CHAPERT VI

BISPECTRUM

As turbulent vortices evolve, they transfer energy from large to small scales. In this case, wave-wave interactions generate the hierarchical turbulent cascade as \( k_1 + k_2 \rightarrow k_3 \). For incompressible flows, under Kolmogorov’s assumptions, we have \( k_1 \approx k_2 = k \) and \( k_3 \approx 2k \). For compressible and magnetized flows, this becomes more complicated and nonlinear wave-wave interactions may take place. Also, we expect MHD turbulence to present more wave modes than on purely hydrodynamical flows and, in this case, the energy cascade can be much more complicated. In order to study this phenomenon, the analysis of the three-point correlation function, or bispectrum, is required (see [Barnett(2002), Masahiro & Bhuvnesh(2004)]).

The bispectrum technique characterizes and searches for nonlinear interactions, which makes it a timely study for interstellar MHD turbulence. However, unlike the standard power spectrum, the bispectrum is a complex function and carries phase information. The power spectrum \( P(k) \) is defined as:

\[
\langle \delta \rho(k_1) \delta \rho(k_2) \rangle = P(k) \delta_D(k_1 + k_2)
\]

where \( \delta_D \) is the Dirac delta function that is zero for all cases but \( k_1 + k_2 = 0 \), the bispectrum \( B_{123} \) is

\[
\langle \delta \rho(k_1) \delta \rho(k_2) \delta \rho(k_3) \rangle = B_{123} \delta_D(k_1 + k_2 + k_3)
\]

As shown above, it is often convenient to think of the bispectrum as it is related to the the power spectrum. The power spectrum is defined as:
\begin{align*}
P(|k|) = \sum_{|k| = \text{const.}} \hat{A}(|k|) \cdot \hat{A}^* (|k|) 
\tag{12}
\end{align*}
and the bispectrum can be defined as:

\begin{align*}
B(|k_1|, |k_2|) = \sum_{|k_1| = \text{const}} \sum_{|k_2| = \text{const}} \hat{A}(|k_1|) \cdot \hat{A}(|k_2|) \cdot \hat{A}^* (|k_1| + |k_2|) 
\tag{13}
\end{align*}

where \( k_1 \) and \( k_2 \) are the wave numbers of two interacting waves, and \( A(k) \) is the original discrete time series data with finite number of elements. In our simulations we take the Fourier transform of each \( A(k) \) in the above equation and average over all frequencies.

In this section we will explore the bispectrum of density and column density for the data cubes. Bispectral analysis can be applied to observational data in a similar way as the synthetic data presented in this paper. Our model does not include self-gravity or external gravity, and only considers the interplay between the gas pressure and magnetic pressure of isothermal gas. By examining the bispectrum of density and comparing it with column densities, we can characterize turbulent flows that would be seen in observational data. We will look at different models of turbulence including pure hydro and different MHD regimes. We will examine differences between the bispectrum of density and column density and discuss what information about the magnetic field might be gained by applying the bispectrum to observable data.

2 Density and Column Density

We ran the bispectrum analysis of the last snapshots of the density cubes, as well as the column density on x and y-directions, for all MHD models. We also performed the bispectrum calculation for a supersonic hydrodynamical simulation. The density and column density bispectra are shown in Figure 9, on the left, center and right columns, respectively. In the the first row we present the results obtained
for the hydrodynamical case followed by the MHD models. The models are labeled for the different turbulent regimes, as the values of $M_A$ and $M_s$. 
Figure 7. Above is the contour analysis of the bispectrum for density and column density and shows the degree of correlation between $k_1$ and $k_2$. The first column shows density, the second shows column density parallel to the magnetic field (x-column density) and the third shows column density perpendicular the the magnetic field (y-column density). Here we compare different sonic and Alfvenic regimes as well as pure hydro turbulence (the top row). Scales are slightly different due to flattening in the bispectrum for a single scale.
All cases show a prominent amplitude of the bispectrum at $k_1 = k_2$, as expected. However, the amplitudes at $k_1 \neq k_2$ are different for each turbulent regime. The more circularly shaped the isocontours are, the more highly correlated the modes are. The hydro models show almost no correlation for $k_1 \neq k_2$ and the distribution is mostly localized in the diagonal line. The MHD models present broader distributions over the $k_1, k_2$ plane.

For the models with same $\mathcal{M}_A$, we found an increase in the wave-wave coupling under the supersonic regime. This phenomenon is expected, as interactions between different scales generate the small scale shocks. Interestingly, comparing models with same sonic Mach number, the sub-Alfvénic ones show increased wave-wave coupling for $k_1 \neq k_2$. Here, non-linear coupling is increased by the magnetic field. For super-Alfvénic models, the large scale (low $k$) magnetic field configuration is destroyed and the MHD modes operate at smaller scales. Both effects reveal that the energy cascade, generated or amplified by $k_1 \neq k_2$ interactions, may work differently at the turbulent regimes, and Kolmogorov scalings may not well-characterize them.

Regarding observable parameters, the column density bispectra do not show so prominent differences compared to the density analysis. However, the supersonic and super-Alfvénic model show a notably different distribution, with larger amplitudes for $k_1 \neq k_2$, similarly to the density bispectrum. A slight amplification is also seen in the subsonic case. Actually, the column density bispectra present larger noise because of the worse statistics ($512^2$ maps compared to the $512^3$ density cube). Therefore, bispectral analysis of observational maps can be used to reveal the turbulent regimes operating in molecular clouds, but only if high resolution maps are available. Interestingly, the broadening of bispectrum distribution by the magnetic field is independent of the orientation of the magnetic field lines regarding the line of sight.
CHAPTER VII

DISCUSSION AND APPLICATIONS

Many surveys provide extensive information on column density statistics. An example of such a project is the Wisconsin H-AlphaMapper (WHAM), whose goal is to provide a survey of H α emission from the ISM over the entire northern sky [Haffner et al.(2003), Hill et al.(2005)]. Observations provide dispersion and emission measurements, which can be interpreted as the column density and the integral of the square of the electron density along the entire line of sight. Surveys such as this supported by numerical studies allow us to better understand the distribution of diffuse plasma in the warm ionized medium. Combining data from different surveys and using the techniques in this paper, it is possible to get valuable insight into processes taking place in different ISM phases.

We find that correlations as well as statistical moments such as skewness and kurtosis are easy to obtain and are very useful in studying MHD turbulence. The moments all have strong dependence on the sonic Mach number. As turbulence becomes more supersonic the skewness of density increases which is agreement with [Kowal, Lazarian & Beresnyak(2007)]. We also see a very rapid growth in kurtosis of density with sonic Mach number. Both column density and density both show increasing asymmetric distributions with increasing $M_s$. If one observes a given skew of column density, the value of $M_s$ can be inferred and related to density. The PDF of density as it depends on sonic Mach number has been examined by [Beresnyak et al.(2005)]. It was found in this study that as sonic Mach number grew, so did the skewness of the PDF, which agrees with our study.

In addition to the moments, correlations give us an idea of what is occurring
in a turbulent system for density clumps as they relate to various quantities. For the correlation of density with magnetic energy, several interesting points are raised in regards to the nature of the interaction between magnetic fields and turbulent gas. Ideally, we have the relationship for the Alfvén speed as \( v_a = \frac{B}{\sqrt{4\pi \rho}} \), thus if the Alfvén speed is the same throughout the model, we should have a \( B = \sqrt{\rho} \) relationship. However, this is not the relationship we see in the correlations in Figure 1. Taking the \( M_a = 0.7 \) and \( M_s = 7.0 \) for example, one can see that the magnetic energy increases less than the density increase due to shocks in the compressed turbulence. Hence, the magnetic energy gets trapped in the clumps and the system does not follow the ideal Alfvén speed, since the magnetic energy cannot increase as fast as the density due to shock waves. Another case to consider is subsonic sub-Alfvénic turbulence. Considering the system locally, there are no clumps for high densities. Locally, the higher density regions push field lines away while the lower density regions allow field lines to move closer together. Thus, as the magnetic energy increases the density gets smaller and most points lie around a mean density. This keeps an equilibrium between the gas pressure and the magnetic pressure. The \( M_a = 2.0 \ M_s = 0.7 \) case is similar to the \( M_a = 0.7 \ M_s = 0.7 \) case but the magnetic energy is 10x smaller.

Figure 2 shows kinetic energy correlated to density. Higher densities have lower kinetic energy, which is a trend that might be expected. The magnetic field seems to speed up density clumps, as can be seen for both densities and in Figures 8 and 9 for column density. For column density with supersonic turbulence, high specific kinetic energies generally reach a critical column density of \( \approx 2 \) and then fall off. The compressibility of densities for supersonic turbulence allows for strong correlations with kinetic energy and prevents scattering due to field lines. The subsonic case does not reach the high densities that characterize supersonic regimes due to lack of shocks. In Figure 3 the correlation of \( M_a \) with density shows different behavior for different models. Supersonic turbulence elongates the structure of the
correlation, and subsonic cases are more circular about the mean than are supersonic. However, strong magnetic fields provide Alfvén shearing which disrupts shock waves [Beresnyak et al.(2005)], making the structure of the correlation even more oblong. Thus a larger range of density and mach number are reached for supersonic turbulence.

Densities and column densities have shown similar characteristics in statistical moments and correlations. In correlations, density and column density are affected in similar manners to compressibility and magnetic field strength. We have shown that density and column densities are further related by the bispectrum and that the bispectrum can describe differences between different turbulent models that could characterize observations.

3 2D and 3D Bispectrum

The bispectrum can give us valuable information regarding how the modes of a nonlinear system correlate. The power spectrum has already be used in several papers to analyze signals, (see for example, [Armstrong, Cordes & B. J. Rickett(1981), Kim & Ryu(2005), Goldman(2000)]). It has been found to be an excellent tool to analyze periodic signals of compressible MHD turbulence. However, it often cannot pick out multiple frequencies as they evolve. In section 5 we analyzed the bispectrum for densities and column densities. Since there are no previous cases of bispectrum being applied for Astrophysical MHD turbulence, this gives us very little material with which to compare results. However, we have found several interesting trends which may provide new methods for both theorists and observers.

The bispectrum of density and column density shown in the contour plot of figure 10 gives information as to how shocks and magnetic fields effect turbulence. It has been shown by [Kowal, Lazarian & Beresnyak(2007)] and [Beresnyak et al.(2005)] that in supersonic turbulence shocks produce compressed
density and a shallower spectrum. Looking at figure 10 for density (first column) it is clear that these shocks play a crucial role in the correlation of modes. The subsonic cases show little correlation between any points except the case of $k_1 = k_2$. The compressed densities from supersonic turbulence are critical for correlations between frequencies since waves will be closer together and therefore have a much higher interaction rate. Also interesting to consider are the differences in the sub-Alfvénic and super-Alfvénic cases for density. We find excellent agreement with [Kowal, Lazarian & Beresnyak(2007), ] in that the density structures are highly related to the presence of magnetic field. When there is a weak magnetic field present it is clear from figure 10 that the system lacks the stronger correlations that are characteristic of the sub-Alfvénic models. For super-Alfvénic simulations correlations in frequency are not as readily made due to large dispersion of density structure. The bispectrum of supersonic hydro models are similar to the super-Alfvénic, supersonic cases. Cases with magnetic field present seem to correlate modes better then cases without magnetic field.

In order to relate to observations, the bispectrum of column densities is compared with density. For both density and column density, the best correlations are found in the case of supersonic sub-Alfvénic. The images for column density are not as resolved as they are for density, and thus column densities have poorer statistics. This is because the data here is 2D and has the resolution of $N^2$ data, and thus statistics are poor compared to densities, which have resolution of $N^3$ data. These images show correlations for all k for both column density parallel and perpendicular to magnetic field. We find the characteristics of MHD turbulence that effect correlations of modes in density such as sub-Alfvénic and supersonic turbulence also effect column density in a similar manner. Because the magnetic field enhances correlations, the bispectrum could be used to characterize the magnetic field in studies similar to [Goodman et al.(1995), Padoan & Nordlund(1999)].
One last point to consider is that our model does not include self-gravity of gas or external gravity, which might pose the questions of how does gravity influence the bispectrum and correlated modes? It has been shown by [Elmegreen & Scalo(2004)] that self-gravity partitions the gas into clouds which contributes to scale-free motions generated by turbulence. The higher density regions will exhibit the most self-gravity. Thus, self-gravity enhances small scale structure and our results should be relatively unaffected, as we are primarily concerned with large scale turbulent cascades.

The bispectrum has proved to be a solid addition to the tools of statistical studies for MHD turbulence. However, it should be noted that very high quality data is required since the multipoint statistics are known to increase noise. For the interstellar medium it has been advised by [Lazarian(1999)] to compare various regions of the sky using the bispectrum to search for like signals. These should be done in future work.
CHAPTER VIII

CONCLUSIONS

In this paper we investigated several different statistics of density structure of compressible MHD turbulence. We examined the skewness and kurtosis of the data as well as studied several different correlations as they related to density. We examined the bispectrum of compressible MHD turbulence, a technique that has been used extensively in the study of large scale structure, yet has never been employed for turbulence. We found:

- Subsonic models are more Gaussian than supersonic models for density and column density distributions due to the presence of shocks. Sonic Mach number can be determined from the skewness of these quantities.

- Correlations of density vs. velocity show no special relationship. A higher magnetic field adds kinetic energy to the gas and speeds up density clumps.

- For both sub-Alfvénic and super-Alfvénic models, the magnetic energy generally increases with increasing density. The magnetic energy seems to become trapped in the density clumps of supersonic turbulence, causing these models to have higher magnetic energies.

- For density vs. $M_a$, the super-Alfvénic cases show larger values for $M_a$ as density increases up to the mean density. For densities larger then the mean density, the values of $M_a$ begin to decrease. For sub-Alfvénic cases we see a very different trend in that $M_a$ increases with increasing density for both supersonic and subsonic cases past the mean density up to a certain critical
density, then rapidly falls off. The subsonic cases see the most rapid increase in $M_a$ with density.

- Column densities show high correlations for supersonic cases and less for subsonic ones due to the compressibility of supersonic turbulence. Sub-Alfvénic cases show less structure than super-Alfvénic ones due to the effects of a magnetic field.

- After applying the bispectrum to MHD turbulence for the first time we find that:

  1. There are strong correlations for cases where $k_1 = k_2$

  2. There are virtually no correlations for $k_1 \neq k_2$ for subsonic cases.

  3. There are correlations with compressible turbulence for all $k_1$ and $k_2$ however, $k_1 = k_2$ remains the strongest.

  4. It is apparent that the introduction of a magnetic field enhances correlations.
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