Motion of Charged Particles

Having obtained Maxwell's equations, before proceeding further we will take a step backwards / laterally and talk briefly about the motion of charged particles in electric and magnetic fields.

This motion will be governed by the Lorentz force law

$$\underline{F} = m\frac{d\underline{v}}{dt} = q(\underline{E} + \underline{v}\wedge\underline{B})$$

In general $\underline{E}$, $\underline{B}$ may be functions of position and time — but for simplicity we will consider only a few special cases

[We consider only non-relativistic velocities and ignore radiation]

* **A. Constant (Static) $\underline{E}$ field ($\underline{B}=0$)**

$$m\underline{a} = m\frac{d\underline{v}}{dt} = q\underline{E}$$

$$\underline{a} = (q/m)\underline{E}$$

Comparison with motion of a particle of mass m under the influence of gravity $\underline{a} = -\underline{g}$ leads us to the conclusion that the path described by the particle will be parabolic in nature.

Assuming $\underline{E} = E\,\underline{1}_z$ then

$$m\frac{dv_x}{dt} = 0 \qquad m\frac{dv_y}{dt} = 0 \qquad m\frac{dv_z}{dt} = qE$$

or $$v_x = \text{constant} = v_{ox} \qquad v_y = \text{constant} = \underline{v_{oy}} \qquad v_z = (q/m)Et + v_{oz}$$

$x = V_{ox}t + x_o$; $y = V_{oy}t + y_o$; $z = \frac{1}{2}(q/m)Et^2 + V_{oz}t + z_o$

|| In other words only the component of velocity along $\underline{E}$ is altered.

* $\underline{B}$ Constant (Static) Magnetic Field ($E=0$)

$$m\underline{a} = m\frac{d\underline{v}}{dt} = q(\underline{v}\wedge\underline{B}) \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

If $\underline{B} = B\hat{z}$ then $m\frac{dv_x}{dt} = qv_yB$, $m\frac{dv_y}{dt} = -qv_xB$

$$m\frac{dv_z}{dt} = 0 \Rightarrow \left.\begin{array}{l} v_z = v_{oz} \\ z = v_{oz}t + z_o \end{array}\right\} \underline{\underline{v} \text{ along } \underline{B} \text{ is unchanged.}}$$

$$m\frac{d^2v_x}{dt^2} = Bq\frac{dv_y}{dt} = -\frac{B^2q^2v_x}{m} \Rightarrow \frac{d^2v_x}{dt^2} = -v_x\frac{B^2q^2}{m^2}$$

assuming @ $t=0$ $\underline{v} = (v_{ox}, 0)$

$v_x^2 + v_y^2 = v_{ox}^2 = $ const. $\Rightarrow$ v in xy plane is constant in magnitude.

$\Rightarrow v_x = v_{ox}\cos(Bq/m)t$ so that $v_y = -v_{ox}\sin\left(\frac{qB}{m}\right)t$

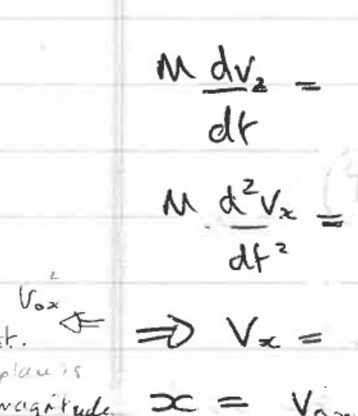
$x = v_{ox}\frac{m}{Bq}\sin\left(\frac{Bq}{m}\right)t + x_o$ $\quad y = y_o + v_{ox}\left(\frac{m}{Bq}\right)\left[\cos\left(\frac{qB}{m}\right)t - 1\right]$

(to ensure @ $t=0$ $y = y_o$)

$\Rightarrow (x-x_o)^2 + \left(y - y_o + v_{ox}\frac{m}{Bq}\right)^2 = \frac{v_{ox}^2 m^2}{B^2q^2}$ — eqn of circle

But the above solution assumes that @ $t=0$ $v_y = 0$ $v_\perp = v_{ox}$

$x_o = y_o = 0$



$\Rightarrow$ particle moves in a circle, radius

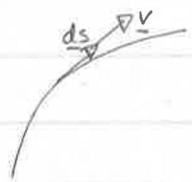
$$\underline{r_c = \frac{v_{ox}m}{Bq}} \quad \text{centred at } \left(x_o, y_o - \frac{v_{ox}m}{Bq}\right)$$

$\Rightarrow$ combined with z motion we have a <u>helix</u>

Using the circumference of the circle in xy plane we obtain τ_c (period) $= \dfrac{2\pi r_c}{V_1} = \dfrac{2\pi V_{ox} M}{Bq\, V_1} = \dfrac{2\pi M}{Bq}$

$V_1 = V_\perp = V_{ox}$

N.B. Work done against / by $\underline{F}_B = \int q(\underline{v} \wedge \underline{B}) \cdot d\underline{s}$ $\quad \left(\int \underline{F}_B \cdot d\underline{s}\right)$



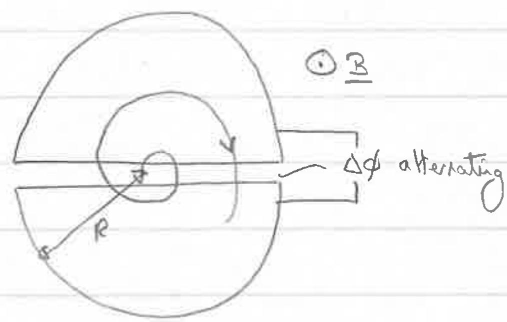
But $d\underline{s} \parallel$ to $\underline{v} \Rightarrow$ Work $= 0$

$\Rightarrow$ KE constant or $\frac{1}{2}MV_{ox}^2 = \frac{1}{2}MV_\perp^2$

(initially we assumed $V_{oy} = 0$)

$\Rightarrow$ $\underline{\tau_c = \dfrac{2\pi M}{Bq}}$ $\qquad$ $\underline{\omega_c = \left(= \dfrac{2\pi}{\tau_c}\right) = \dfrac{Bq}{M}}$ (cyclotron frequency) (independent of r_c)

Cyclotron



$\odot \underline{B}$ $\qquad$ For $q > 0$

Particle is accelerated across the gap, gaining energy (increasing velocity). Increased velocity causes it to move to a larger radius. Next time it reaches the gap the potential has reversed, thus accelerating the particle again, moving it to even larger radius

$\frac{Mv^2}{R} = qvB \Rightarrow$ $\quad (KE)_{MAX} = \frac{1}{2}Mv^2 = \dfrac{M}{2}\dfrac{B^2q^2R^2}{M^2} = \dfrac{B^2q^2R^2}{2M}$

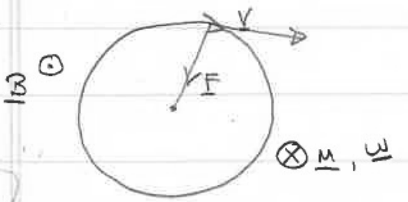
In practise because of the relativistic increase of mass for light particles ($e^\pm$) the technique is more difficult (ω_c not constant)

It is worth noting that a charged particle moving in a circular path has a magnetic moment $\underline{m}_\mu$

$$= \left(\frac{M}{q}\right) \frac{\tfrac{1}{2} M v_\perp^2}{B^2} \underline{\omega}_c = - \frac{\tfrac{1}{2} M v_\perp^2 \underline{B}}{B^2}$$

$$\underline{M}_\mu = I \pi r_c^2 = \frac{q \pi v_\perp^2 M^2}{\tau_c B^2 q^2} = \frac{\pi v_\perp^2 M^2 B q}{q B^2 2\pi M} = \frac{M v_\perp^2}{2B} = \frac{\tfrac{1}{2} M v_\perp^2}{B}$$

For e+



$\underline{F} = M\frac{d\underline{v}}{dt} = q(\underline{v}_\perp \wedge \underline{B})$

Kinematics / mechanics $\Rightarrow \underline{\omega} = -\frac{\underline{B}q}{M}$ parallel

$\frac{d\underline{v}_\perp}{dt} = \underline{\omega} \wedge \underline{v}_\perp$

Application of r.h. rules leads us to the vector equation

$$\underline{M}_\mu = - \frac{\tfrac{1}{2} M v_\perp^2}{B^2} \underline{B}$$

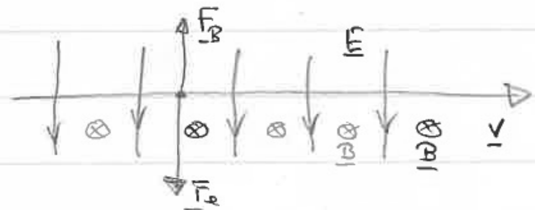
(Read p 537 – example of motion under non-uniform $\underline{B}$ – leading to magnetic reflection, useful in attempts to control particles in fusion reactors)

* C Constant (static) $\underline{E}$ and $\underline{B}$ fields

Of course inclusion of arbitrary $\underline{E}$ and $\underline{B}$ fields will make the solution for the motion more complex.

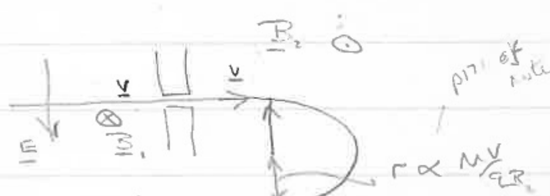
Two cases will be considered

(i) "Crossed" $\underline{E}$ and $\underline{B}$ fields as shown.



If the particle moves with velocity $\underline{v}$ at right angles to $\underline{E}$ and $\underline{B}$. Then there will be no deflection when $\underline{F}_B = \underline{F}_E$

$$q(\underline{v} \wedge \underline{B}) = q\underline{E}$$
$$v = E/B$$

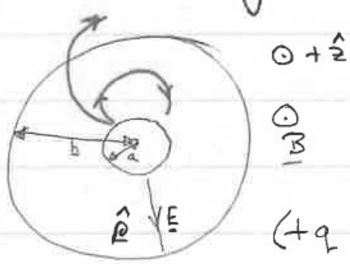


This is the principle of the mass spectrometer.

e.g. Select particles of known $\underline{v}$ then apply a 2nd $\underline{B}$ field alone $r \propto M/q$

Aim: To measure $\underline{B}$ field.
Inner cylinder heated filament produces e^-
Condition no current between inner + outer cylinder obtained by adjusting $\Delta\phi$ (potential between plates).
[Note: this analysis for charge q not e^-]

(ii) The magnetron



(+q shown)

Two conducting cylinders, co-axial with an $\underline{E}$ field between them. $\underline{B}$ along axis of cylinder as shown. Charged particles (+) originating on the inner cylinder will follow trajectories as shown, depending on the strength of the $\underline{B}$ field.

For a particle starting at rest on the inner cylinder we may write

$$\tfrac{1}{2}m(v_\rho^2 + v_\phi^2) + q\phi(\rho) = q\,\phi(a)$$

where v_ρ, v_ϕ are radial and tangential velocities and ϕ is the potential.

$$\Rightarrow \quad \tfrac{1}{2}m\left[\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2\right] = q\left[\phi(a) - \phi(\rho)\right]$$

For a particle which just grazes the outer cylinder, at that position $\rho = b$ and $\left(\frac{d\rho}{dt}\right) = 0$

$$\Rightarrow \quad b^2\left(\frac{d\phi}{dt}\right)^2_{\text{at }\rho=b} = \frac{2q\Delta\phi}{m} \qquad (**)$$

But $\quad \underline{F} = m\underline{a} = q\left[\underline{E} + (\underline{v}\wedge\underline{B})\right]$

$$\underline{B} = B\hat{\underline{z}} \qquad \underline{v} = v_\rho\hat{\underline{\rho}} + v_\phi\hat{\underline{\phi}}$$

$$\Rightarrow \quad \underline{v}\wedge\underline{B} = -\hat{\underline{\phi}}Bv_\rho + Bv_\phi\hat{\underline{\rho}}$$

$$\Rightarrow \quad ma_\phi = -qBv_\rho \qquad (*)$$

$\hat{\rho}(\rho, \phi, z)$ $\quad \frac{d\hat{\rho}}{dt} = \frac{d\phi}{dt}\underbrace{\frac{\partial\hat{\rho}}{\partial\phi}}_{\hat{\phi}} + \frac{\partial\rho}{\partial t}\frac{\partial\hat{\rho}}{\partial\rho} + \frac{\partial z}{\partial t}\frac{\partial\hat{\rho}}{\partial z}$

Also $\underline{v} = v_\rho \hat{\rho} + v_\phi \hat{\phi}$

$$\Rightarrow \frac{d\underline{v}}{dt} = \frac{dv_\rho}{dt}\hat{\rho} + v_\rho \frac{d\hat{\rho}}{dt} + \frac{dv_\phi}{dt}\hat{\phi} + v_\phi \frac{d\hat{\phi}}{dt}$$

$$= \frac{dv_\rho}{dt}\hat{\rho} + v_\rho \frac{d\phi}{dt}\frac{d\hat{\rho}}{d\phi} + \frac{dv_\phi}{dt}\hat{\phi} + v_\phi \frac{d\phi}{dt}\frac{d\hat{\phi}}{d\phi}$$

$$= \frac{dv_\rho}{dt}\hat{\rho} + v_\rho \frac{d\phi}{dt}\hat{\phi} + \frac{dv_\phi}{dt}\hat{\phi} + v_\phi \frac{d\phi}{dt}(-\hat{\rho})$$

$$\Rightarrow a_\phi = v_\rho \frac{d\phi}{dt} + \frac{dv_\phi}{dt} = v_\rho \frac{d\phi}{dt} + \frac{d}{dt}\left(\rho \frac{d\phi}{dt}\right)$$

$$= v_\rho \frac{d\phi}{dt} + \frac{d\rho}{dt}\frac{d\phi}{dt} + \rho \frac{d^2\phi}{dt^2}$$

$$= \rho \frac{d^2\phi}{dt^2} + 2\left(\frac{d\rho}{dt}\right)\left(\frac{d\phi}{dt}\right)$$

From $*$

$$\Rightarrow m a_\phi \rho = m \frac{d}{dt}\left[\rho^2 \frac{d\phi}{dt}\right] = -qBv_\rho \rho = -qB\frac{d}{dt}\left(\frac{\rho^2}{2}\right)$$

$$\Rightarrow \rho^2 \frac{d\phi}{dt} = -\frac{qB}{2m}\rho^2 + \text{const}$$

Using $\rho = a$ when $(d\phi/dt) = 0$

$$\rho^2 \frac{d\phi}{dt} = -\frac{qB}{2m}(\rho^2 - a^2)$$

therefore at $\rho = b$ $\quad b^2\left(\frac{d\phi}{dt}\right)_b = -\frac{qB}{2m}(b^2 - a^2)$

$(**)$

Substituting in our original relationship yields

$$B^2 = \frac{8mb^2\Delta\phi}{q(b^2-a^2)^2}$$

This allows us to evaluate B when $\Delta\phi$ is such that

There is <u>no current</u> measured between the cylinders
(see problem A9)

* Suggest you read about the betatron (A-4)

Problems – Appendix A : 1, 3, 5, 9, 11

Similar to Magnetron