11. Special Methods in Electrostatics

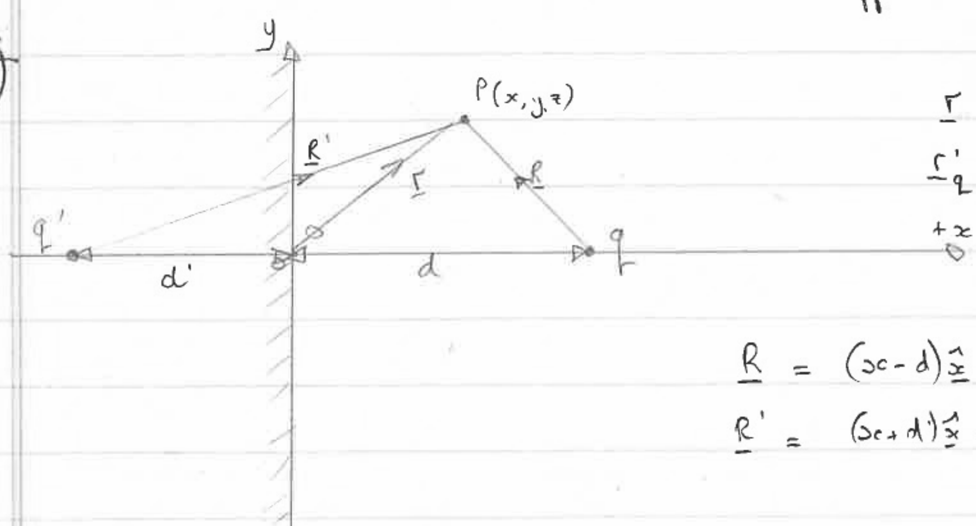
Our last topics in electrostatics will be to introduce two different methods of obtaining ϕ in certain situations where it would be impossible (or at minimum extremely difficult) to use the equation.

$$\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\underline{r}')\,d\tau'}{R}$$

A. Method of Images

This is an extremely limited method, but provides a quick solution in the cases to which it can be applied.

e.g.



$\underline{r} = \hat{\underline{x}}x + \hat{\underline{y}}y + \hat{\underline{z}}z$

$\underline{r}'_q = \hat{\underline{x}}d$

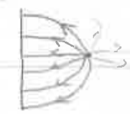
$\underline{r}'_{q'} = -\hat{\underline{x}}d'$

$\underline{R} = (x-d)\hat{\underline{x}} + \hat{\underline{y}}y + \hat{\underline{z}}z$

$\underline{R}' = (x+d')\hat{\underline{x}} + \hat{\underline{y}}y + \hat{\underline{z}}z$

A charge $+q$ sits a distance d in front of a grounded ($\phi = 0$) conductor. The conductor occupies the <u>whole</u> yz plane. We know that for $x < 0$ $\phi = 0$ everywhere. What is the potential ϕ for $x > 0$. The presence of the conductor will affect the $\underline{E}$ field of the single charge $+q$ so we

But we know "intuitively" that the E field will look like



Cannot use $\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\underline{r}')\,d\tau'}{R}$.

In the method of images we try to find a set of "image" charges whose effect is to simulate the behaviour of the infinite conductor.

i.e.

$$\phi_{TRUE} = \sum_{actual} \frac{q}{4\pi\varepsilon_0 R} + \sum_{image} \frac{q_i}{4\pi\varepsilon_0 R_i}$$

Since the image charges will simulate the behaviour of the conductor they are located in the conductor — <u>outside</u> the region where we are trying to find ϕ.

On the surface of the conductor $\phi(0, y, z) = 0$ which is the boundary condition any image charge must satisfy.

For the solution of this problem, try a charge q' a distance d' from the origin on the x-axis, within the conductor, as shown; then

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R} + \frac{q'}{R'}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left[(x-d)^2 + y^2 + z^2\right]^{1/2}} + \frac{q'}{\left[(x+d')^2 + y^2 + z^2\right]^{1/2}}\right]$$

but $\phi(0, y, z) = 0$

$$\Rightarrow \frac{q}{(d^2 + y^2 + z^2)^{1/2}} = -\frac{q'}{(d'^2 + y^2 + z^2)^{1/2}}$$

$\Rightarrow$ $\underline{q' = -q}$ and $\underline{d' = d}$ Uniqueness Theorem

Thus $$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left[(x-d)^2 + y^2 + z^2\right]^{1/2}} - \frac{1}{\left[(x+d)^2 + y^2 + z^2\right]^{1/2}}\right]$$

is the solution for $x > 0$ (remember for $x < 0$ $\phi = 0$).

The $\underline{E}$ field for $x > 0$ can be found from $\underline{E} = -\underline{\nabla}\phi$

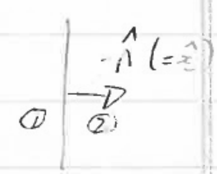
$$E_x = -\frac{\partial \phi}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left[\frac{x-d}{\left[(x-d)^2+y^2+z^2\right]^{3/2}} - \frac{x+d}{\left[(x+d)^2+y^2+z^2\right]^{3/2}}\right]$$

$$E_y = -\frac{\partial \phi}{\partial y} = \frac{qy}{4\pi\varepsilon_0}\left[\frac{1}{\left[(x-d)^2+y^2+z^2\right]^{3/2}} - \frac{1}{\left[(x+d)^2+y^2+z^2\right]^{3/2}}\right]$$

$E_z = E_y$ (with z substituted for y in multiplier at front)

with which we can verify that the boundary conditions for $\underline{E}$ are satisfied.

$(\hat{\underline{n}} = \hat{\underline{x}})$ at $x = 0$ (on the conductor surface) [Tangential components are zero @ $x = 0$ by inspection]



$$\hat{\underline{n}} \cdot \underline{E}_v = -\frac{qd}{2\pi\varepsilon_0\left[d^2+y^2+z^2\right]^{3/2}} \qquad \text{in vacuo}$$

$$\hat{\underline{n}} \cdot \underline{E}_c = 0 \qquad \text{in conductor}$$

$E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$

$$\hat{\underline{n}} \cdot (\underline{E}_v - \underline{E}_c) = \sigma/\varepsilon_0 \qquad \text{b.c.'s on normal } \underline{E} \text{ component}$$

$$\Rightarrow \quad \sigma = -\frac{qd}{2\pi\left(d^2+y^2+z^2\right)^{3/2}}$$

(this charge has been <u>induced</u> on the conductor by the charge $+q$)

Total charge on conductor is given by

$$Q = -\frac{qd}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dy\,dz}{\left(d^2+y^2+z^2\right)^{3/2}} = -q$$

as expected – the image charge is $-q$.

The force on q should be given by

$$\underline{F}_q = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{\underline{x}} \qquad \text{if the image method is correct.}$$

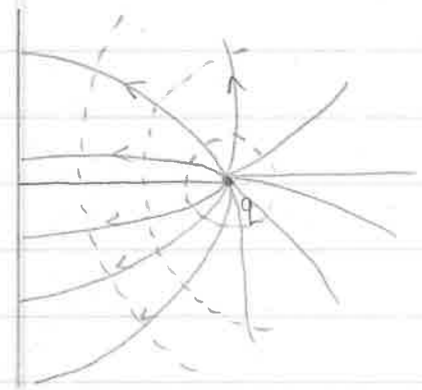
To verify this what is E_x at q? $(d,0,0)$

$$E_x = -\frac{q}{4\pi\varepsilon_0}\left[\frac{2d}{((2d)^2)^{3/2}}\right] = -\frac{q}{16\pi\varepsilon_0 d^2}$$

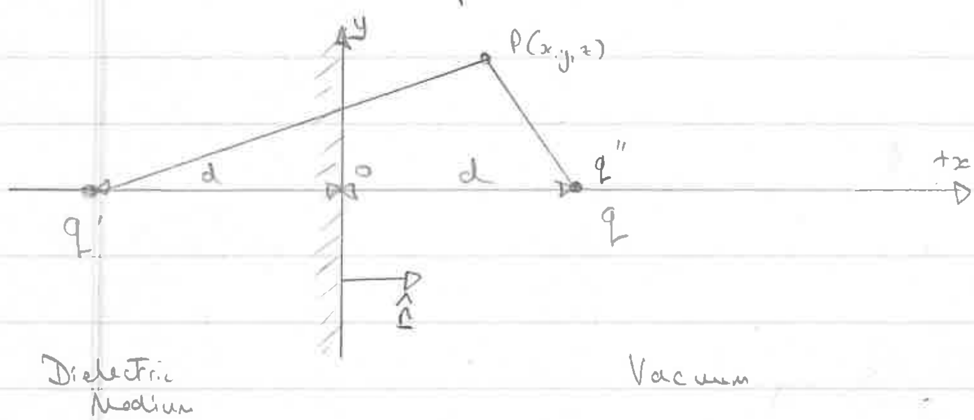
[Where we have ignored the first term of E_x (E_y, E_z) since this is the E due to the charge itself]

so that $\underline{F}_q = qE_x\hat{\underline{x}} = -\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\underline{x}}$ as expected

Finally we expect the equipotential and $\underline{E}$ field lines to be as shown below.



* As a second example, imagine that instead of a conductor we have a semi-infinite dielectric medium (l.i.h)



The first thing to note is that we must try to evaluate ϕ in two regions $x > 0$ (vacuum) and $x < 0$ (dielectric medium).

For vacuum take a guess at q' positioned at $x = -d$

then $$\phi_V = \frac{1}{4\pi\varepsilon}\left[\frac{q}{[(x-d)^2+y^2+z^2]^{1/2}} + \frac{q'}{[(x+d)^2+y^2+z^2]^{1/2}}\right]$$

and $$\underline{E_{Vx}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{(x-d)q}{[(x-d)^2+y^2+z^2]^{3/2}} + \frac{(x+d)q'}{[(x+d)^2+y^2+z^2]^{3/2}}\right]$$

$$\underline{E_{Vy}} = \frac{y}{4\pi\varepsilon_0}\left[\frac{q}{[(x-d)^2+y^2+z^2]^{3/2}} + \frac{q'}{[(x+d)^2+y^2+z^2]^{3/2}}\right]$$

similarly for E_{Vz}.

For the dielectric medium let us guess (!) q'' at $x = +d$ $(y = z = 0)$ will do the trick.

$$\phi_D = \frac{1}{4\pi\varepsilon_0}\left[\frac{q''}{[(x-d)^2+y^2+z^2]^{1/2}}\right]$$

and $$E_{Dx} = \frac{q''}{4\pi\varepsilon_0}\frac{(x-d)}{[(x-d)^2+y^2+z^2]^{3/2}} \qquad E_{Dy} = \frac{q''y}{4\pi\varepsilon_0}\frac{1}{[(x-d)^2+y^2+z^2]^{3/2}}$$

similarly for E_{Dz}

To check whether our guesses are correct — do they satisfy the boundary conditions at $x = 0$?

(i) Normal (x) components $[D_{Vn} - D_{In} = \sigma_f]$ $\underline{D} = \varepsilon\underline{E}$

$$K_V\varepsilon_0 E_{Vx} - K_D\varepsilon_0 E_{Dx} = \sigma_{free}\ (=0)$$

$$\Rightarrow E_{Vx} = K_D E_{Dx}$$

or $$\cancel{\frac{1}{4\pi\varepsilon_0}}\left[\frac{-dq}{\cancel{(d^2+y^2+z^2)^{3/2}}} + \frac{dq'}{\cancel{(d^2+y^2+z^2)^{3/2}}}\right] = \frac{-K_D\, q''d}{\cancel{4\pi\varepsilon_0(d^2+y^2+z^2)^{3/2}}}$$

$$-q + q' = -K_D q''$$

(ii) Tangential components (yz)

$$E_{Vy} = E_{Dy} \qquad E_{Vz} = E_{Dz}$$

$$\frac{y}{\cancel{4\pi\varepsilon_0}}\left[\frac{q}{\cancel{(d^2+y^2+z^2)^{3/2}}} + \frac{q'}{\cancel{(d^2+y^2+z^2)^{3/2}}}\right] = \frac{q''y}{\cancel{4\pi\varepsilon_0[d^2+y^2+z^2]^{3/2}}}$$

$$q + q' = q''$$

$$\Rightarrow \quad 2q = q''(1 + K_D) \qquad q'' = \frac{2q}{(1+K_D)}$$

and $$q' = -\frac{(K_D - 1)}{(K_D + 1)} q$$

Thus the charges q q' and q'' are equivalent to q and the semi infinite di-electric.

Force on q $$\underline{F}_q = \frac{q q' \hat{\underline{x}}}{4\pi \varepsilon_0 4d^2} = -\left(\frac{K_D - 1}{K_D + 1}\right) \frac{q^2}{16\pi\varepsilon_0 d^2} \hat{\underline{x}}$$

and is negative as expected. q is attracted by induced charge on di-electric.

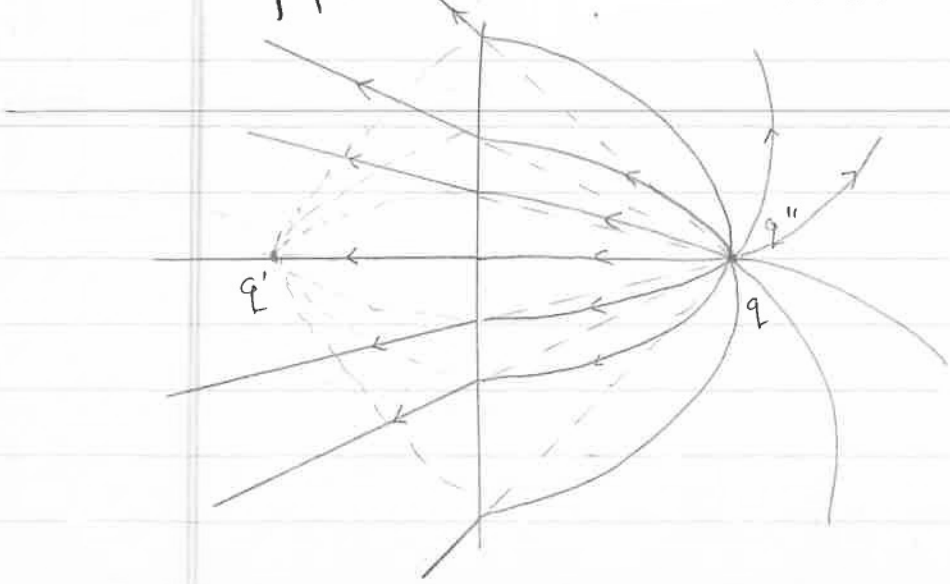
The bound charge density σ_b on the surface is given by ($\hat{\underline{n}} = \hat{\underline{x}}$)

$$\sigma_b = \underline{P} \cdot \hat{\underline{n}} = P_{Dx} = (K_D - 1)\varepsilon_0 E_{Dx}(0, y, z) \qquad \underline{P} = (K_D - 1)\varepsilon_0 \underline{E}$$

$$\sigma_b = -\frac{(K_D - 1)\varepsilon_0}{(1 + K_D)} \frac{2qd}{4\pi\varepsilon_0 (d^2 + z^2 + y^2)^{3/2}}$$

which could be integrated over the whole surface to give q'

Lines of force are indicated below, together with the "imaginary" lines which account for the "method of images" description. Equipotentials can easily be constructed.



* The example of the point charge and the grounded conducting sphere and the point charge and the insulated uncharged conducting sphere are left for your "edification" (p176 - 180). Suffice to say the method of images is very ad-hoc - and is only useful in some very limited cases.

As another "method" it is sometimes possible to adapt previous solutions such that they "represent" new (different) problems. (Sect 11.3).

* However, when all else fails we may have to resort to solving Laplace's equation or Poisson's equation.

$$\nabla^2 \phi = 0 \qquad\qquad \nabla^2 \phi = -\rho/\varepsilon_0$$

[Remember $\oint_S \underline{E} \cdot d\underline{a} = \sum_{in} q/\varepsilon_0 \qquad \underline{E} = -\underline{\nabla}\phi$

$$\Rightarrow \oint_S -\underline{\nabla}\phi \cdot d\underline{a} = -\int_V \underline{\nabla} \cdot \underline{\nabla}\phi \, d\tau = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau$$

$$\underline{\nabla}^2 \phi = -\rho/\varepsilon_0$$]

Once again - analytic solutions are possible in only a few very symmetric cases - but non-analytic solutions (obtained by computational methods) will always exist (for physical - "real" - problems).

Laplace/Poisson's equation is specific to E+M, but very a similar form appears as Schrodinger's equation in QM

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi$$

Therefore it will not hurt to look at the simple solutions of Laplace / Poisson's equation.

In either case and in any co-ordinate system the solution proceeds via separation of variables.

i.e. in x, y, z we assume $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

or in r, θ, ϕ " " $\phi(r,\theta,\varphi) = R(r).\,\Theta(\theta).\,\Phi(\varphi)$

$\phi(\rho,\varphi,z) = P(\rho).\,\Phi(\varphi).\,Z(z)$

[careful with notation ϕ is voltage (potential) φ is azimuthal angle!]

Rectangular co-ordinate solution and example is easier than (r,θ,ϕ) [p185 → 190], therefore we will proceed with the spherical case

$$\nabla^2 \phi = 0 \qquad \text{Laplace eqn.}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

Use $\phi = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

then
$$\frac{1}{R}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \frac{1}{\Theta}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right)\right] = -\frac{1}{\Phi}\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}\right]$$

Multiply both sides by $r^2\sin^2\theta$ then r.h.s is a function only of φ, l.h.s only of r, θ ⇒ they must equal the same constant

$$\Rightarrow \quad \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = -\xi^2$$

$$\Phi = \beta_1 \cos(\xi\varphi + \beta_2) \qquad \text{is a solution}$$

$$(\text{or } \beta_1 e^{i(\xi\varphi+\beta_2)})$$

with $\beta_1 \beta_2$ ξ constants.

the l.h.s of the equation then becomes

$$\frac{1}{R}\left[\sin^2\Theta \frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \frac{1}{\Theta}\left[\sin\Theta \frac{\partial}{\partial \Theta}\left(\sin\Theta\frac{\partial \Theta}{\partial \Theta}\right)\right] = \xi^2$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] = -\frac{1}{\Theta}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right)\right] + \frac{\xi^2}{\sin^2\Theta}$$

Once again l.h.s and r.h.s are functions of a single variable $\Rightarrow$ both sides must be equal to the same constant _ k

$$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] = k = -\frac{1}{\Theta}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right)\right] + \frac{\xi^2}{\sin^2\Theta}$$

R equation gives

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0$$

Which has a solution of the form $R = \alpha r^l$, then

$$r^2 l(l-1)\alpha r^{l-2} + 2rl\alpha r^{l-1} - k\alpha r^l = 0$$

$\Rightarrow$ $\underline{l(l+1) = k}$ (assuming $R \neq 0$)

therefore Θ equation becomes

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right) - \frac{\xi^2\,\Theta}{\sin^2\Theta} + l(l+1)\Theta = 0$$

there are two important cases to consider

a) $\underline{\xi = 0}$: $\Rightarrow$ axial symmetry, no φ dependence of the solution. $[\phi(\Theta, r)]$

the R equation is unchanged but the Θ equation is simplified to

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right) + l(l+1)\Theta = 0$$

the solutions of which are Legendre polynomials (see multi-pole expansion discussion chapter 8).

$$\Theta_\ell(\theta) = P_\ell(\cos\theta)$$

where

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

and higher order functions can be obtained from the recursion relation

$$\ell(\ell+1)\,P_{\ell+1}(\cos\theta) = (2\ell+1)\cos\theta\,P_\ell(\cos\theta) - \ell\,P_{\ell-1}(\cos\theta)$$

[Note that physical solutions only exist for $\ell = 0,1,2,3\ldots$ – no negative ℓ's and ℓ must be integer]

b) $\xi \neq 0$: there is a ϕ dependence (not covered by this text)

the Θ equation is of the more general form

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \left[\ell(\ell+1) - \frac{\xi^2}{(1-\cos^2\theta)}\right]\Theta = 0$$

which can also be solved in terms of Legendre polynomials of the form $P_{\ell\xi}(\cos\theta)$

‖ ξ is more commonly written as m, and you may see the direct analogy between QM + EM. In QM ℓ, m are the orbital angular momentum and z component of orbital angular momentum quantum numbers. ‖

For this situation the Θ and Φ solutions are usually combined in the form of spherical harmonic functions

$$Y_{\ell m} = \Theta_{\ell m}\,\Phi_m$$

Where
$$Y_{10} = \left(3/4\pi\right)^{1/2} \cos\theta \qquad (\propto P_1(\cos\theta))$$
$$Y_{1+1} = -\left(3/8\pi\right)^{1/2} \sin\theta\, e^{+i\phi}$$
$$Y_{1-1} = \left(3/8\pi\right)^{1/2} \sin\theta\, e^{-i\phi}$$
$$Y_{20} = \underbrace{\left(5/16\pi\right)^{1/2}}_{\text{normalisation constants}} (3\cos^2\theta - 1) \qquad (\propto P_2(\cos\theta))$$

so that $\displaystyle\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}\, Y_{lm} \sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'}$

(orthogonality condition on Y)

* Returning now to the R equation
$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$
and substituting $R = \alpha r^n$ we find
$$r^2 n(n-1)\alpha r^{n-2} + 2r\alpha n r^{n-1} - l(l+1)\alpha r^n = 0$$
$$n^2 - n + 2n - l(l+1) = 0$$
$$n(n+1) = l(l+1)$$
$$\Rightarrow \quad n = l \quad \text{or} \quad n = -(l+1)$$
thus a general solution for R may be written
$$R_l(r) = A_l r^l + B_l r^{-(l+1)}$$
where the constants A_l, B_l are specific to a particular l.

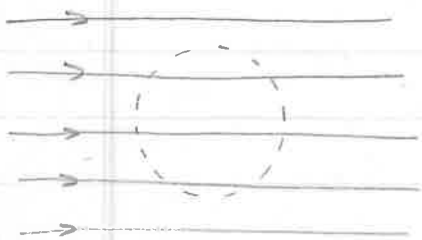
* The general solution to Laplace's equation in spherical co-ordinates may therefore be written as
$$\phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(A_l r^l + B_l r^{-(l+1)}\right) Y_{lm}(\theta,\phi)$$

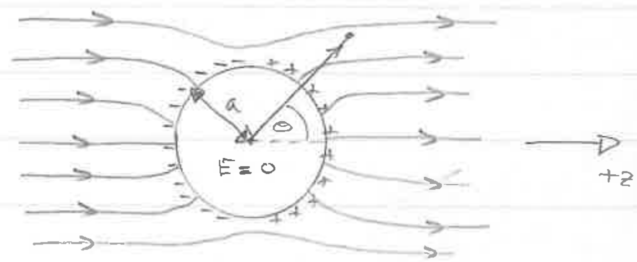
or for the axially symmetric case $(m=0)$

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

* Examples of solving Laplace's equation in axially symmetric cases. [Remember Laplace's equation is $\nabla^2\phi = 0$, which means there is no free charge density].

A. Grounded conducting sphere in uniform external $\underline{E}$ field





Since at the surface of a conductor we can have no tangential $\underline{E}$ field [$E_{in\,tan} = E_{out\,tan}$, but $E_{in\,tan} = 0$], the distortion of the external $\underline{E}$ field must look something like above.

In general we will use the general solution above and with the help of boundary conditions determine the A_ℓ, B_ℓ separately in the different regions defined by the problem.

These bc's are defined by the values of ϕ at $r=\infty$, $r=0$ and on the "boundaries" of the problem. (in this case $r=a$)

Outside sphere (at large r) $\quad \underline{E} = E_0\hat{\underline{z}}$ and since $\underline{E} = -\underline{\nabla}\phi$

$$E_0 = -\frac{\partial\phi}{\partial z} \qquad \left[\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0\right]$$

$$\Rightarrow \quad \phi = -E_0 z = -E_0 r\cos\theta$$

[Integration constant we set to be zero; note $\phi(\infty) \neq 0$]

b.c.'s are

$$\phi_{out}(a,\Theta) = 0 = \phi_{in} \quad \text{(sphere grounded)}$$
$$\phi(\infty,\Theta) = -E_0 r\cos\Theta \quad (\text{uniform } \underline{E} \text{ @ } r=\infty)$$
$$\left[\text{plus } \phi(r,\Theta) = 0 \text{ all } r<a \Rightarrow \phi_{inside} = 0\right]$$

Substituting the first b.c. into the general solution gives

$$\sum_{l=0}^{\infty}\left(A_l a^l + B_l a^{-l-1}\right)P_l(\cos\Theta) = 0$$

It can easily be shown that if $\sum_{l=0}^{\infty} C_l P_l(\cos\Theta) = 0$
then all the $C_l = 0$
(using the orthogonality relation for the P_l $\int_0^{\pi} P_l(\cos\Theta)P_{l'}(\cos\Theta)\sin\Theta\, d\Theta = \delta_{ll'}\frac{2}{2l+1}$)

$$\Rightarrow \quad A_l a^l + B_l a^{-l-1} = 0$$
$$B_l = -a^{2l+1}A_l.$$

Using $\phi(\infty,\Theta) = -E_0 r\cos\Theta$

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\Theta) = -E_0 r\cos\Theta$$

$$(A_1+E_0)\,r\,P_1(\cos\Theta) + \sum_{l\neq 1}^{\infty} A_l r^l P_l(\cos\Theta) = 0$$

for the same reason as above we must have

$$A_1 = -E_0 \quad \text{and } \underline{\text{all}} \text{ other } A_l = 0$$

$$\Rightarrow \quad B_1 = +a^3E_0 \quad \text{and } \underline{\text{all}} \text{ other } B_l = 0$$

so that

$$\phi(r,\Theta) = -E_0 r\cos\Theta + \frac{a^3E_0\cos\Theta}{r^2} \quad (r\geq a)$$

(uniform field has picked out $(\cos\Theta)$ form in general solution)

$$\phi(r, \theta) = 0 \qquad r < a$$

For $r > a$ the first term for ϕ is simply due to the external field. The second term has the dipole form where

$\phi = \frac{p \cos\theta}{4\pi\varepsilon_0 r^2}$

$$p = 4\pi\varepsilon_0 a^3 E_0$$

i.e. the sphere has become polarised, the surface charge density creating the dipole moment can be found from the normal component of $\underline{E}$ on the surface $(E = -\nabla\phi)$

$$E_r = -\frac{\partial\phi}{\partial r} = E_0 \cos\theta \left(1 + 2a^3/r^3\right)$$

$E_{2n} - E_{1n} = \sigma/\varepsilon_0$

$$E_r(r = a) = 3E_0 \cos\theta$$

b.c.'s for $\underline{E}$ on surface (normal component) $\Rightarrow$ $\sigma/\varepsilon_0 = E_{\text{norm out}} - E_{\text{norm in}}$

$$= 3E_0\cos\theta - 0$$

$$\underline{\sigma = 3\varepsilon_0 E_0 \cos\theta}$$

the total charge $\int \sigma \, da = 0$ (as required - polarisation merely redistributes charge)

and the tangential component of $\underline{E}$ $\quad E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -E_0\sin\theta\left(1 - a^3/r^3\right)$

which when $r = a$ becomes zero, thus satisfying the b.c. for $\underline{E}$ tangential. [E_ϕ is zero since there is no ϕ dependence]

B. Dielectric Sphere in Uniform External $\underline{E}$ field

Similar to case A above, except that for $r < a$ the potential will not be zero.

B.C.'s on ϕ, where ϕ_o $(r > a)$ $\quad \phi_i$ $(r < a)$

$$\phi_o(\infty, \theta) = -E_0 r \cos\theta \qquad (1)$$

$$\phi_i(0,\theta) \quad \text{finite} \qquad (2)$$

$\underline{E}$ tangential continuous at discontinuity

(E_θ tangential) $$-\frac{1}{r}\left(\frac{\partial \phi_o}{\partial \theta}\right) = -\frac{1}{r}\left(\frac{\partial \phi_i}{\partial \theta}\right) \qquad (3) \quad r = a$$

$\underline{E}$ normal $\hat{n}\cdot\left(\kappa_o \varepsilon_o \underline{E}_o - \kappa_i \varepsilon_o \underline{E}_i\right) = \sigma_f$ (but $\sigma_f = 0$)

(E_r normal) $$-\varepsilon_o\left(\frac{\partial \phi_o}{\partial r}\right) = -\kappa_i \varepsilon_o \frac{\partial \phi_i}{\partial r} \qquad (4) \quad r = a$$

ϕ continuity gives no new information (same condition as (3) gives) [No azimuthal dependence assumed at start of problem]

Because of the similarity to the previous example we can write

$$\phi_o = \left(-A_o r + \frac{B_o}{r^2}\right)\cos\theta \qquad (r > a)$$

(the sphere will become polarised in the external field).
Inside the sphere, due to the $\underline{E}$ normal boundary condition it is reasonable to expect (cos θ will appear on ϕ_o, assume it is also on ϕ_i)

$$\phi_i = \left(-A_i r + \frac{B_i}{r^2}\right)\cos\theta \qquad (r < a)$$

B.C. (1) $r \to \infty$ $\quad \phi_o = -E_o r\cos\theta$

$$\Rightarrow A_o = E_o$$

B.C. (2) $r \to 0$ $\quad \phi_i \to$ finite

$$\Rightarrow B_i = 0$$

Substituting ϕ_o and ϕ_i into B.C.'s (3) and (4) gives

$$-E_o a + B_o/a^2 = -A_i a \; ; \; -E_o - 2B_o/a^3 = -\kappa_i A_i$$

which on solution give

$$A_i = \frac{3E_o}{(\kappa_i + 2)} \; ; \; B_o = \frac{(\kappa_i - 1)}{(\kappa_i + 2)} a^3 E_o$$

and

Dielectric Sphere in a Uniform E Field.

b.c.'s

$$\phi_o(\infty, \theta) = -E_o r\cos\theta \quad (1)$$

$$\phi_i(0, \theta) \quad \text{finite} \quad (2)$$

$$\frac{\partial \phi_o}{\partial \theta}(r=a) = \frac{\partial \phi_i}{\partial \theta}(r=a) \quad (3) \quad E_{tang}$$

$$\frac{\partial \phi_o}{\partial r}(r=a) = \kappa\frac{\partial \phi_i}{\partial r}(r=a) \quad (4) \quad E_{norm}$$

$$\phi_o(r=a) = \phi_i(r=a) \quad (5) \quad \phi \text{ continuous}$$

General solutions:

$$\phi_i = \sum \left(A_{il} r^l + B_{il} r^{-(l+1)}\right) P_l(\cos\theta)$$

$$\phi_o = \sum \left(A_{ol} r^l + B_{ol} r^{-(l+1)}\right) P_l(\cos\theta)$$

b.c. (2) $\Rightarrow$ all $B_{il} = 0$

b.c. (1) ($@\ r \to \infty$)

$$-E_o r\cos\theta = \sum A_{ol} r^l P_l(\cos\theta)$$

Using the condition

$$\sum C_l P_l(\cos\theta) = 0$$

$$\Rightarrow \text{all } C_l = 0$$

[proved using orthogonality of complete set of functions $P_l(\cos\theta)$]

then $A_{o1} = -E_o$

and all other $A_{ol} = 0$

$$\Rightarrow \quad \phi_o = -E_o r\cos\theta + \sum B_{ol} r^{-(l+1)} P_l(\cos\theta)$$

$$\phi_i = \sum A_{il} r^l P_l(\cos\theta)$$

Using bc (5) at $r = a$

$$-E_0 a P_1(\cos\theta) + \sum B_{0\ell}\, a^{-(\ell+1)} P_\ell(\cos\theta) = \sum A_{i\ell}\, a^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \sum_{\substack{\text{not}\\ \ell=1}} \left[A_{i\ell}\, a^\ell - B_{0\ell}\, a^{-(\ell+1)}\right] P_\ell(\cos\theta) + \left(E_0 a + A_{i1} a - B_{01}/a^2\right) P_1(\cos\theta) = 0 \quad (6)$$

Using (4) at $r = a$

$$-E_0\cos\theta + \sum -(\ell+1) B_{0\ell}\, a^{-(\ell+2)} P_\ell(\cos\theta) = \sum \ell A_{i\ell}\, a^{\ell-1} P_\ell(\cos\theta)\, \kappa$$

$$\Rightarrow \sum_{\substack{\text{not}\\ \ell=1}} \left[\kappa \ell A_{i\ell}\, a^{\ell-1} + (\ell+1) B_{0\ell}\, a^{-(\ell+2)}\right] P_\ell(\cos\theta) + \left(E_0 + A_{i1}\kappa + 2B_{01}/a^3\right) P_1(\cos\theta) = 0 \quad (7)$$

Using if $\sum C_\ell P_\ell(\cos\theta) = 0$ then all $C_\ell = 0$

From (6) $\quad A_{i\ell}\, a^\ell = B_{0\ell}\, a^{-(\ell+1)} \quad - (8)$

and $\quad E_0 a + A_{i1} a - B_{01}/a^2 = 0 \quad - (9)$

from (7) $\quad \kappa \ell A_{i\ell}\, a^{\ell-1} + (\ell+1) B_{0\ell}\, a^{-(\ell+2)} = 0 \quad - (10)$

$\quad E_0 + A_{i1}\kappa + 2B_{01}/a^3 = 0 \quad - (11)$

Substituting (8) into (10) $\quad A_{i\ell} = B_{0\ell}\, a^{-2\ell-1}$

$$\kappa \ell B_{0\ell}\, a^{-\ell-2} + (\ell+1) B_{0\ell}\, a^{-\ell-2} = 0$$

$$\left.\begin{array}{l}\Rightarrow B_{0\ell} = 0\\ \Rightarrow A_{i\ell} = 0\end{array}\right\} \text{ for all } \ell \text{ except } \ell = 1$$

Finally (9) and (11) give

$$\left.\begin{aligned} -E_0 a + B_{01}/a^2 &= A_{i1}\, a \\ -E_0 - 2B_{01}/a^3 &= \kappa A_{i1} \end{aligned}\right\}$$

can be solved to give

$$A_{i1} = \frac{-3E_0}{(\kappa+2)}$$

and

$$B_{01} = \frac{(\kappa-1)a^3 E_0}{(\kappa+2)}$$

Note: If $\sum C_\ell P_\ell(\cos\theta) = 0$

and $\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \delta_{\ell\ell'} \frac{2}{2\ell+1}$

(orthogonality of complete set of functions $P_\ell(\cos\theta)$)

then

$$\sum_{\ell=0}^{\infty} \int_0^\pi C_\ell P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = 0$$

$$\frac{C_{\ell'}\, 2}{(2\ell+1)} = 0 \quad \Rightarrow \quad C_{\ell'} = 0$$

i.e. all $C_\ell = 0$

So that

$$\phi_0 = -E_0 r\cos\theta + \frac{(\kappa-1)}{(\kappa+2)}\frac{a^3}{r^2} E_0\cos\theta \qquad (r > a)$$

and

$$\phi_i = \frac{-3E_0 r\cos\theta}{(\kappa+2)} = -\frac{3E_0\, z}{(\kappa+2)} \qquad (r < a)$$

therefore

$$\phi_o = -E_o r\cos\Theta + \left(\frac{\kappa_1-1}{\kappa_1+2}\right)\frac{a^3}{r^2}E_o\cos\Theta \qquad r>a$$

$$\phi_i = \frac{-3E_o r\cos\Theta}{(\kappa_1+2)} = \frac{-3E_o z}{(\kappa_1+2)} \qquad r<a$$

$$\underline{E}_i = -\underline{\nabla}\phi_i = \frac{3}{(\kappa_1+2)}\underline{E}_o$$ smaller than $\underline{E}_o$ but parallel to it.

$$\underline{P} = (\kappa_1-1)\varepsilon_o\underline{E}_i = \frac{3(\kappa_1-1)\varepsilon_o}{(\kappa_1+2)}\underline{E}_o$$

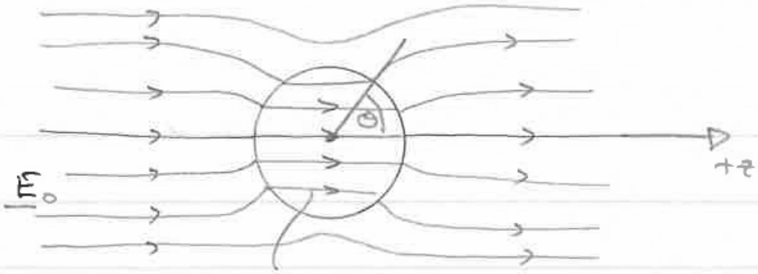
total dipole moment of sphere $p = \frac{4\pi a^3}{3}|\underline{P}| = 4\pi a^3\varepsilon_o\frac{(\kappa_1-1)}{(\kappa_1+2)}E_o$

which means

$$\phi_o = \underbrace{-E_o r\cos\Theta}_{\text{original } E_o \text{ field}} + \underbrace{\frac{p\cos\Theta}{4\pi\varepsilon_o r^2}}_{\text{dipole created by polarisation of sphere}}$$

p 196 (11-131) and following takes this example somewhat further introducing $\underline{E}_{loc}$ – field within the dielectric caused by the bound charges which are induced on its surface. I will finish this section with the remark that if $\kappa_1 \to \infty$ we obtain the same results as case A – the conducting sphere $\Rightarrow$ for electrostatics a conductor acts as a dielectric with $\kappa = \infty$.

And finally the $\underline{E}$ field picture of our dielectric sphere.



$$\underline{E}_i = \underline{E}_0 + \underline{E}_{LOC} \quad (\underline{E}_{LOC} \propto -\underline{P})$$

* Finally in this chapter a simple example of Poisson's equation where ϕ is a function only of r.

$$\nabla^2 \phi = -\rho/\varepsilon_0$$

Sphere with uniform charge density ρ $(r<a)$ $[\rho = 0 \;\; r>a]$

For $\underline{r>a}$ $(\rho = 0)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 0$$

$$\Rightarrow \quad \phi_2 = A_2 + B_2/r$$

For $\underline{r<a}$ $(\rho = \rho)$

$$\frac{d}{dr}\left(r^2 \frac{d\phi_1}{dr}\right) = -\frac{\rho r^2}{\varepsilon_0}$$

$$\Rightarrow \quad \phi_1 = -\frac{\rho r^2}{6\varepsilon_0} + A_1 + \frac{B_1}{r}$$

B.C.'s

$$\phi_2(\infty) = 0 \quad \Rightarrow \quad A_2 = 0$$

$$\phi_1(0) \text{ finite} \quad \Rightarrow \quad B_1 = 0$$

ϕ continuous $\qquad \phi_2(a) = \phi_1(a)$

E_{norm} continuous $(\sigma_{\text{on surface}} = 0)$ $\qquad -(\partial\phi_1/\partial r)_{r=a} = -(\partial\phi_2/\partial r)_{r=a}$

$$\Rightarrow \quad \frac{B_2}{a} = -\frac{\rho a^2}{6\varepsilon_0} + A_1 \;\; ; \;\; \frac{\rho a}{3\varepsilon_0} = \frac{B_2}{a^2}$$

$$\Rightarrow B_2 = (\rho a^3/3\varepsilon_0) \quad \text{and} \quad A_1 = \frac{\rho a^2}{3\varepsilon_0} + \frac{\rho a^2}{6\varepsilon_0} = \frac{\rho a^2}{2\varepsilon_0}$$

$\therefore$

Sect 5.2 eqn (5-22) $\phi_2 = \frac{\rho a^3}{3\varepsilon_0}\frac{1}{r}$ $(r > a)$ outside sphere $\left[= \frac{Q}{4\pi\varepsilon_0 r} \right]$

" eqn (5-23) $\phi_1 = -\frac{\rho r^2}{6\varepsilon_0} + \frac{\rho a^2}{2\varepsilon_0} = \frac{\rho}{6\varepsilon_0}(3a^2 - r^2)$ $(r < a)$ inside sphere.

N.B. It is important to realise that this result is exactly the same as was obtained from the expression for the scalar potential

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho' d\tau'}{R}$$

There is no new physics here, just an alternative method.

Problems – Chapter 11 : 3, 5, 9, 13 (Use p185 + Chap 7) — images

17 (long), 25 (long), 27 (incomplete) — Laplace

31 — Poisson.