21. Maxwell's Equations

The story so far:

$\underline{\nabla} \cdot \underline{D} = \rho_f$ — Coulombs Law / Gauss' Law

$\underline{\nabla} \wedge \underline{E} = -\partial \underline{B}/\partial t$ — Faraday's Law of induction

$\underline{\nabla} \cdot \underline{B} = 0$ — No magnetic monopoles (from Ampere's law)

$\underline{\nabla} \wedge \underline{H} = \underline{J}_f$ — Ampère's Law

As you probably know, this is not the final form of Maxwell's equations.

$$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{H}) = \underline{\nabla} \cdot \underline{J}_f = 0$$ (div curl of any vector is zero)

but equation of continuity (charge conservation) $\underline{\nabla} \cdot \underline{J}_f + \frac{\partial \rho_f}{\partial t} = 0$

Thus, unless $\partial \rho_f / \partial t = 0$ there is something wrong with the fourth equation, above.

The "fix" – due to Maxwell – is the introduction of the displacement current $\underline{J}_d \Rightarrow \underline{\nabla} \wedge \underline{H} = \underline{J}_f + \underline{J}_d$

then

$$\underline{\nabla} \cdot \underline{\nabla} \wedge \underline{H} = 0 = \underline{\nabla} \cdot \underline{J}_f + \underline{\nabla} \cdot \underline{J}_d = -\frac{\partial \rho_f}{\partial t} + \underline{\nabla} \cdot \underline{J}_d$$

$$\Rightarrow \quad \underline{\nabla} \cdot \underline{J}_d - \partial/\partial t (\underline{\nabla} \cdot \underline{D}) = 0$$

or $\underline{\nabla} \cdot \left(\underline{J}_d - \partial \underline{D} / \partial t \right) = 0$

$\underline{J}_d = \dfrac{\partial \underline{D}}{\partial t}$ displacement current [due to rate of change of displacement vector]

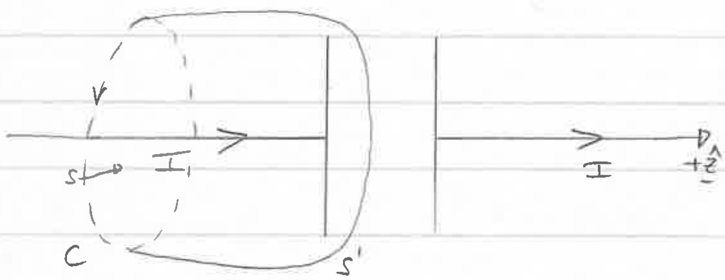
and so $\underline{\nabla} \wedge \underline{H} = \underline{J}_f + \dfrac{\partial \underline{D}}{\partial t}$

The inclusion of the displacement current leaves the tangential boundary conditions on $\underline{H}$ (defined by $\underline{\nabla} \wedge \underline{H}$) unchanged, but alters the integral form of Ampere's Law $H_{2n} - H_{1n} = \underline{K}_f \wedge \hat{\underline{n}}$

$$\oint_C \underline{H} \cdot d\underline{s} = I_{f\,encl} + I_{d\,encl}$$

$$\int_S \underline{J}_f \cdot d\underline{a} + \int_S \frac{\partial \underline{D}}{\partial t} \cdot d\underline{a}$$

* Practically the need for the displacement current can be seen by examining the parallel plate capacitor [Immediately following connection to a battery]



Evaluate $\oint_C \underline{H} \cdot d\underline{s} = \int_S \underline{J}_f \cdot d\underline{a}$

When S is the disk bounded by C $\int_S \underline{J}_f \cdot d\underline{a} = I$

But S can be chosen at will – provided that it is bounded by C – that is we could choose S'. In this case $\int_{S'} \underline{J}_f \cdot d\underline{a}$ is zero. But since $\underline{D} = \varepsilon_0 \underline{E} = q/A\ \hat{\underline{z}}$ [between the plates of the capacitor]

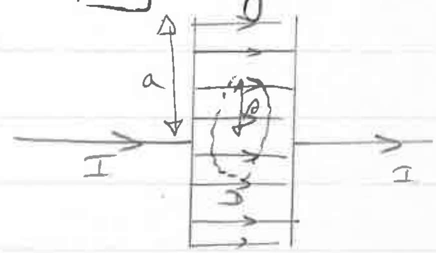
$E = \Delta\phi / d \qquad C = q / \Delta\phi = A\varepsilon_0 / d \qquad$ or $\oint_S \underline{D} \cdot d\underline{a} = q$

then $\partial D/\partial t = (\partial q/\partial t)/A = I/A$

and $\int_{S'} \frac{\partial D}{\partial t} \cdot da = I = \int_{S'} J_d \cdot da$

and inclusion of the displacement current saves the day.
[Note that D is only changing when the charge on the capacitor plates is changing (i.e. during charge or discharge (DC)). When the capacitor is no longer being charged there is neither I_f or I_d]

The displacement current is distributed uniformly throughout the space occupied by the capacitor, thus when applying ampere's law around a circular loop as shown we obtain



$$\oint H \cdot ds = \frac{I \cdot \pi \rho^2}{\pi a^2}$$

* Application of the boundary conditions on H leads to a radial surface current K_f flowing outwards from the centre of the capacitor on each plate. [p352]
[This is the distribution of charge during the Δt when the current is turned on].

* It must be emphasised that I_d, is not a current of moving charges — but it does provide an alternative, equivalent, source of magnetic field H.

$$\nabla \wedge H = \frac{\partial D}{\partial t} \left(+ J_f \right)$$

Can be thought of as the "magnetic" form of Faraday's Law of induction — a changing E (D) gives rise to an H (B) field. (Just as Faraday's Law says that a changing $B \Rightarrow E$)

In fact, we will see later that the displacement current term of $\nabla \wedge \underline{H}$ is <u>essential</u> to the existence of em waves.

* After inclusion of the displacement current we obtain the complete set of Maxwell's equations

$\nabla \cdot \underline{D} = \rho_f$ Coulomb / Gauss $\oint_S \underline{D} \cdot d\underline{a} = \int_V \rho_{fr} \, d\tau$

$\nabla \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ Faraday $\oint_C \underline{E} \cdot d\underline{s} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$

$[\mathcal{E} = -d\Phi_B/dt]$

$\nabla \cdot \underline{B} = 0$ Ampere + no mag monopoles $\oint_S \underline{B} \cdot d\underline{a} = \Phi_B = 0$

[note that Φ_B not necessarily zero unless surface is closed]

$\nabla \wedge \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$ Ampere + Maxwell $\oint_C \underline{H} \cdot d\underline{s} = \int_S \underline{J}_f \cdot d\underline{a} + \int_S \underline{J}_d \cdot d\underline{a}$

Add to these

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$$

$$\underline{H} = \underline{B}/\mu_0 - \underline{M}$$

the Lorentz force law $\underline{F} = q(\underline{E} + \underline{v} \wedge \underline{B})$

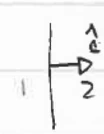
and the boundary conditions

$$D_{2n} - D_{1n} = \sigma_f$$

$$E_{2t} - E_{1t} = 0$$

$$B_{2n} - B_{1n} = 0$$

$$H_{2t} - H_{1t} = \underline{K}_f \wedge \hat{\underline{n}} \quad [\hat{n} \text{ from } 1 \to 2]$$



plus equation of continuity $\nabla \cdot \underline{J} + \partial \rho / \partial t = 0$

and we have the entire classical em theory in one simple, compact, form.

* Depending on the particular circumstances it may be more convenient to write Maxwell's equations in slightly different forms e.g. $\underline{E}, \underline{B}$ $(+\underline{P}, \underline{M})$ or for l.i.h. media in terms of just $\underline{E}, \underline{B}$ or $\underline{E}, \underline{H}$. (p 354-356) [and HW problems]

[$\underline{D} = \varepsilon \underline{E}$, $\underline{B} = \mu \underline{H}$ are called constitutive equations]

* Energy Flow and the Poynting Vector

$$W = \int_V \underline{J}_f \cdot \underline{E} \, d\tau = \text{rate at which em energy is converted into heat (watts)}$$

(see Chap 12) 12-35

Substituting $\underline{J}_f = \underline{\nabla} \wedge \underline{H} - \partial \underline{D}/\partial t$

$$W = \int \underline{E} \cdot (\underline{\nabla} \wedge \underline{H}) \, d\tau - \int_V (\underline{E} \cdot \partial \underline{D}/\partial t) \, d\tau$$

but

$$\underline{E} \cdot (\underline{\nabla} \wedge \underline{H}) = \underline{H} \cdot (\underline{\nabla} \wedge \underline{E}) - \underline{\nabla} \cdot (\underline{E} \wedge \underline{H}) = -\underline{H} \cdot \partial \underline{B}/\partial t - \underline{\nabla} \cdot (\underline{E} \wedge \underline{H})$$

$$\Rightarrow \quad W = -\int_V \left(\underline{H} \cdot \partial \underline{B}/\partial t + \underline{E} \cdot \partial \underline{D}/\partial t \right) d\tau - \int_V \underline{\nabla} \cdot (\underline{E} \wedge \underline{H}) \, d\tau$$

but for l.i.h media $\underline{H} = \underline{B}/\mu$ and $\underline{D} = \varepsilon \underline{E}$

$$\Rightarrow \quad W = -\frac{\partial}{\partial t} \int_V \underbrace{\left(\tfrac{1}{2} \varepsilon \underline{E}^2 + B^2/2\mu \right)}_{\text{total em energy}} d\tau - \oint_S (\underline{E} \wedge \underline{H}) \cdot d\underline{a}$$

$u_{tot} = u_e + u_m$

Poynting's Theorem

$$\Rightarrow \quad -\frac{\partial}{\partial t} \int_V u_{tot} \, d\tau = W + \oint_S (\underline{E} \wedge \underline{H}) \cdot d\underline{a}$$

rate of decrease of em energy = rate of conversion to heat + rate of loss of em energy from V (rate at which energy leaves V. If < 0 then energy is flowing into volume)

$\underline{E} \wedge \underline{H} = \underline{S}$ — Poynting vector (power flux)
energy current density, "points" in the direction of energy flow.
Watts/m^2 (intensity)

Problems : Chap 21 — 1, 3, 7, 9, 12

↓

lin, non-homogeneous
$\Rightarrow \varepsilon(\underline{r}) \quad \mu(\underline{r})$

$\varepsilon \underline{E} = D \qquad B = \mu(r) H$