22. Scalar And Vector Potentials

* Maxwell's equations do not <u>explicitly</u> involve the scalar and vector potentials (ϕ, $\underline{A}$)

(M1) $$\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \underline{\nabla} \cdot \underline{P}\right) \qquad [\underline{\nabla} \cdot \underline{D} = \rho_f]$$

(M2) $$\underline{\nabla} \wedge \underline{E} = -\partial \underline{B}/\partial t$$

(M3) $$\underline{\nabla} \cdot \underline{B} = 0$$

$$[\underline{\nabla} \wedge \underline{H} = \underline{J}_f + \partial \underline{D}/\partial t]$$

(M4) $$\underline{\nabla} \wedge \underline{B} = \mu_0\left(\underline{J}_f + \underline{\nabla} \wedge \underline{M} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t} + \frac{\partial \underline{P}}{\partial t}\right)$$

Eqns 21.20 + 21.33

(remember $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$
$\underline{H} = \underline{B}/\mu_0 - \underline{M}$)

But it is sometimes convenient to write Maxwell's equations in terms of potentials – from which the fields can be found, if required. This method is useful <u>particularly when considering</u> time varying fields – i.e. $\underline{E}$, $\underline{B}$, and the potentials can be functions of $\underline{r}$ and t. [e.g. Radiation treatment Chap 28]

* Since $\underline{\nabla} \cdot \underline{B} = 0$ we can write

$$\underline{B}(\underline{r}, t) = \underline{\nabla} \wedge \underline{A}(\underline{r}, t) \qquad [\underline{\nabla} \cdot \underline{\nabla} \wedge \underline{A} = 0 \text{ always}]$$

The scalar potential ϕ was introduced because $\underline{\nabla} \wedge \underline{E} = 0$ in electrostatics $[\underline{\nabla} \wedge \underline{\nabla}\phi = 0]$

but $$\underline{\nabla} \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\frac{\partial}{\partial t}(\underline{\nabla} \wedge \underline{A}) = -\underline{\nabla} \wedge \partial \underline{A}/\partial t$$

or $$\underline{\nabla} \wedge \left(\underline{E} + \partial \underline{A}/\partial t\right) = 0$$

therefore we introduce a <u>new</u> scalar potential ϕ where

$$-\underline{\nabla}\phi = \underline{E} + \frac{\partial \underline{A}}{\partial t}$$

so that $\underline{\nabla} \wedge \underline{\nabla}\phi = 0$

and

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$$

Together with $\underline{B} = \underline{\nabla} \wedge \underline{A}$ we can now substitute back into the first and last of Maxwells equation giving

$$\underline{\nabla}.\underline{E} = \tfrac{1}{\varepsilon_0}(\rho_f - \underline{\nabla}.\underline{P})$$

(1) — $$-\nabla^2\phi - \tfrac{\partial}{\partial t}(\underline{\nabla}.\underline{A}) = +\tfrac{1}{\varepsilon_0}(\rho_f - \underline{\nabla}.\underline{P})$$

and

$$\underline{\nabla} \wedge \underline{B} = \mu_0\left(\underline{J}_f + \underline{\nabla}\wedge\underline{M} + \varepsilon_0 \tfrac{\partial \underline{E}}{\partial t} + \tfrac{\partial \underline{P}}{\partial t}\right)$$

$$\underline{\nabla}\wedge\underline{\nabla}\wedge\underline{A} = \mu_0 \underline{J}_f + \mu_0 \underline{\nabla}\wedge\underline{M} - \mu_0\varepsilon_0 \underline{\nabla}\frac{\partial\phi}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2 \underline{A}}{\partial t^2} + \mu_0 \tfrac{\partial \underline{P}}{\partial t}$$

$$\underline{\nabla}(\underline{\nabla}.\underline{A}) - \underline{\nabla}^2\underline{A} = \text{"}$$

(2) — $$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\frac{\partial^2\underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\underline{J}_f - \mu_0\underline{\nabla}\wedge\underline{M} - \mu_0\frac{\partial \underline{P}}{\partial t}$$

In principle (1) and (2) could be solved for $\underline{A}$ and ϕ from which we use $\underline{\nabla}\wedge\underline{A} = \underline{B}$ and $\underline{E} = -\underline{\nabla}\phi - \partial\underline{A}/\partial t$ to determine the fields $\underline{E}$ and $\underline{B}$.

Of course in any general case this will be extremely difficult, since $\underline{A}$ and ϕ appear in both equations and $\underline{P}$ and $\underline{M}$ are potentially functions of $\underline{E}, \underline{B}$ (hence $\underline{A}$ and ϕ).

* Before proceeding to a more specific case it is

Worth noting

(i) $\underline{A}$ and ϕ remain continuous at a surface of discontinuity in properties. (even though they may now both be functions of time)

(ii) When $\underline{A}$, ϕ are not dependent on time (1) and (2) reduce to the familiar forms

Poisson's equation

$$\nabla^2\phi = -\rho/\varepsilon_0 = -\tfrac{1}{\varepsilon_0}(\underbrace{\rho_f - \underline{\nabla}\cdot\underline{P}}_{\rho_b})$$

$$\text{and } \underline{\nabla}^2\underline{A} = -\mu_0\underline{J} = -\mu_0(\underline{J}_f + \underbrace{\underline{\nabla}\wedge\underline{M}}_{J_m})$$

(using $\underline{\nabla}\cdot\underline{A} = 0$) p252 (16-17)

*

Specialising to the case of l.i.h materials where $\underline{J}_f = \sigma\underline{E} + \underline{J}_f'$

↳ current sources other than conductivity e.g. particle beams

$M = \chi H$
$= \frac{(\kappa_m - 1)B}{\mu_0\kappa_m}$

$$\underline{P} = \underline{E}\,\varepsilon_0(\kappa_e - 1)$$

$$\underline{M} = \frac{\underline{B}}{\mu_0}(1 - 1/\kappa_m)$$

$$(1) \Rightarrow \nabla^2\phi + \underline{\nabla}\cdot\partial\underline{A}/\partial t = -1/\varepsilon_0(\rho_f - \underline{\nabla}\cdot\underline{P}) = -\rho/\varepsilon_0$$

but $\rho = \rho_f/\kappa_e$ (10-59)

$$= \frac{-\rho_f}{\kappa_e\varepsilon_0}$$

$$= -\rho_f/\varepsilon$$

$$(1N) \leftarrow \nabla^2\phi + \underline{\nabla}\cdot\partial\underline{A}/\partial t = -\rho_f/\varepsilon$$

$$(2) \Rightarrow \underline{\nabla}^2\underline{A} - \mu_0\varepsilon\frac{\partial^2 A}{\partial t^2} - \underline{\nabla}(\underline{\nabla}\cdot\underline{A} + \mu_0\varepsilon_0\,\partial\phi/\partial t)$$

$$= -\mu_0\underline{J}_f' - \mu_0\sigma\underline{E} - \underline{\nabla}\wedge\underline{B}(1 - 1/\kappa_m) - \mu_0\frac{\partial E}{\partial t}(\kappa_e - 1)\varepsilon_0$$

but $\underline{\nabla} \wedge \underline{A} = \underline{B}$ and $\underline{E} = -\underline{\nabla}\phi - \partial \underline{A}/\partial t$

$\Rightarrow$

$$\underline{\nabla} \wedge \underline{B} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{A} \qquad \frac{\partial \underline{E}}{\partial t} = -\underline{\nabla}\left(\partial\phi/\partial t\right) - \partial^2 \underline{A}/\partial t^2$$
$$= \underline{\nabla}(\underline{\nabla}.\underline{A}) - \underline{\nabla}^2\underline{A}$$

substituting in (2)

$$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\,\partial^2 \underline{A}/\partial t^2 - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\,\partial\phi/\partial t\right) = -\mu_0 \underline{J}_f'$$
$$- \mu_0\sigma\left(-\underline{\nabla}\phi - \partial \underline{A}/\partial t\right) - \left[\underline{\nabla}(\underline{\nabla}.\underline{A}) - \underline{\nabla}^2\underline{A}\right]\frac{(\kappa_m - 1)}{\kappa_m}$$
$$+ \mu_0\varepsilon_0(\kappa_e - 1)\left[\underline{\nabla}\left(\partial\phi/\partial t\right) + \partial^2 \underline{A}/\partial t^2\right]$$

$$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\kappa_m\,\partial^2 \underline{A}/\partial t^2 - \mu_0\varepsilon_0(\kappa_e - 1)\,\partial^2 \underline{A}/\partial t^2\,\kappa_m - \mu_0\kappa_m\sigma\,\partial \underline{A}/\partial t$$
$$- \underline{\nabla}\left(\kappa_m\underline{\nabla}.\underline{A} + \kappa_m\mu_0\varepsilon_0\,\partial\phi/\partial t - \underline{\nabla}.\underline{A}(\kappa_m - 1) + \kappa_m\mu_0\sigma\phi + \mu_0\varepsilon_0(\kappa_e - 1)\kappa_m\,\frac{\partial\phi}{\partial t}\right)$$
$$= -\mu_0 \underline{J}_f'\kappa_m$$

or

$$\underline{\nabla}^2\underline{A} - \mu\varepsilon\,\partial^2 \underline{A}/\partial t^2 - \mu\sigma\,\partial \underline{A}/\partial t - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu\varepsilon\,\partial\phi/\partial t + \mu\sigma\phi\right) = -\mu \underline{J}_f' \qquad \text{(2N)} \leftarrow$$

(1N) can also be written

$$\nabla^2\phi - \underline{\mu\varepsilon}\,\partial^2\phi/\partial t^2 - \underline{\mu\sigma}\,\partial\phi/\partial t + \partial/\partial t\left(\underline{\nabla}.\underline{A} + \underline{\mu\varepsilon}\,\partial\phi/\partial t + \underline{\mu\sigma\phi}\right) = -\rho_f/\varepsilon$$

where the terms underlined have been "added and subtracted"

(1N) and (2N) have a very similar form, but each equation still involves both variables $\underline{A}$ and ϕ – they are not <u>separated</u>.

But $\underline{A}$ is a vector field, the Helmholtz theorem tells us that $\underline{A}$ is not defined until we know the form of both $\underline{\nabla} \wedge \underline{A}$ and $\underline{\nabla} \cdot \underline{A}$.

[Go to optional proof of Helmholtz theorem HT-1]

$\underline{\nabla} \wedge \underline{A}$ is defined as $\underline{B}$. Since there are no "a priori" conditions on $\underline{\nabla} \cdot \underline{A}$ we will define

$$\underline{\nabla} \cdot \underline{A} = -\mu\varepsilon \, \partial\phi/\partial t - \mu\sigma\phi$$

— the Lorentz condition

then (2N) and (1N) are simplified [cf $\underline{\nabla} \cdot \underline{A} = 0$ in Chap 16]

$$\nabla^2\phi - \mu\sigma\frac{\partial\phi}{\partial t} - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_f}{\varepsilon}$$

$$\underline{\nabla}^2\underline{A} - \mu\sigma\frac{\partial \underline{A}}{\partial t} - \mu\varepsilon\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu\underline{J}_f'$$

and separated!

If the form of ρ_f and $\underline{J}_f'$ are known we can, in principle, solve for $\underline{A}$ and ϕ and hence obtain $\underline{E}$ and $\underline{B}$.

* If the medium under consideration is non conducting (a vacuum provides a good example!) then $\sigma = 0 \Rightarrow \underline{J}_f = \sigma\underline{E} + \underline{J}_f' = \underline{J}_f'$

and we obtain

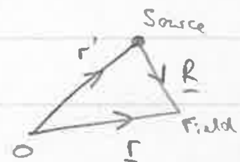
The Helmholtz Theorem

p37 Sect (1-20)

What do we need to specify completely a vector field (eg. $\underline{E}$ or $\underline{B}$)?

* The Helmholtz theorem says that if we know the divergence and the curl of a vector field then the vector field is defined uniquely.

$\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \wedge \underline{F}$ are called the sources of the field, and are functions of the source (primed) co-ordinates only



$$\underline{F} = -\underline{\nabla}\phi + \underline{\nabla} \wedge \underline{A}$$

Where

$$\phi = \frac{1}{4\pi}\int_{V'} \frac{(\underline{\nabla}\cdot\underline{F})'}{|\underline{R}|} d\tau' \qquad \underline{A} = \frac{1}{4\pi}\int_{V'} \frac{(\underline{\nabla}\wedge\underline{F})'}{|\underline{R}|} d\tau'$$

(V' finite volume)

* The "proof" is as follows

$$\underline{\nabla}\cdot\underline{F} = -\nabla^2\phi = -\frac{1}{4\pi}\int_{V'} \nabla^2 \left(\frac{\underline{\nabla}\cdot\underline{F}}{|\underline{R}|}\right)' d\tau'$$

$$= -\frac{1}{4\pi}\int_{V'} (\underline{\nabla}\cdot\underline{F})' \, \nabla^2\left(1/|\underline{R}|\right) d\tau'$$

now $\nabla^2\left(1/|\underline{R}|\right)$

$$= \underline{\nabla}\cdot\underline{\nabla}(1/\underline{R})$$

$$= -\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right)$$

but $\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\,\delta^3(\underline{R})$

GO TO DD1

Aside : The dirac delta function

$$\text{If } \underline{v} = \frac{\hat{\underline{R}}}{R^2} \text{ then } \underline{\nabla}\cdot\underline{v} = \nabla\cdot\frac{\hat{\underline{R}}}{R^2} \overset{(1-44)}{=} -\underline{\nabla}\cdot\nabla\left(\frac{1}{R}\right) = -\nabla^2\left(\frac{1}{R}\right) = 0$$

(when $R \neq 0$)

(1-145) ($R \neq 0$)

$$\text{but } \int_V \underline{\nabla}\cdot\underline{v}\, d\tau = \oint_S \underline{v}\cdot d\underline{a}$$

$$= \oint_S \frac{\hat{\underline{R}}\cdot d\underline{a}}{R^2}$$

Assume that S is the surface of a sphere centred at the origin ($R=0$) then $d\underline{a} = \hat{\underline{R}}\, R^2 \sin\theta\, d\theta\, d\phi$ and

$$\oint_S \frac{\hat{\underline{R}}\cdot\hat{\underline{R}}\, R^2 \sin\theta\, d\theta\, d\phi}{R^2} = 4\pi$$

(for a sphere of any radius) $\Downarrow$ all contribution is from $\underline{R} = 0$

$$\text{or } \int_V (\underline{\nabla}\cdot\underline{v})\, d\tau = 4\pi$$

but $\underline{\nabla}\cdot\underline{v} = 0$?!! (here we exclude the possibility $R=0$)

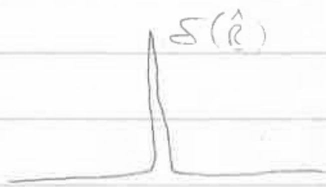
The problem is that at $R=0$ $\underline{v} \to \infty$ and when evaluating $\underline{\nabla}\cdot\underline{v}$ we were implicitly dividing by zero. But the surface integral has no such problems so we accept the result

$$\int_V (\underline{\nabla}\cdot\underline{v})\, d\tau = 4\pi$$

and introduce the <u>dirac delta function</u> $\delta(R)$ such that

$\underline{\nabla}\cdot\underline{v} = 0$ everywhere except at $R=0$ (origin)

$$\underline{\nabla}\cdot\underline{v} = \underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\,\delta^3(\underline{R})$$



$$\text{where } \int_{V'} \delta^3(\underline{R})\, d\tau' = 1$$

(all space)

For our purposes the important fact is that

$$\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\,\delta^3(\underline{R})$$

Also $$\int_{V'} F(r)\, \delta^3(\hat{\underline{R}})\, d\tau' = F(r')$$

$$\underline{R} = \underline{r} - \underline{r}'$$

$F(r)\,\delta^3(\hat{\underline{R}})$ is zero everywhere except $\hat{\underline{R}} = 0$
$\Rightarrow \underline{r} = \underline{r}'$

$\Rightarrow$

$$F(r') \underbrace{\int_{V'} \delta^3(\hat{\underline{R}})\, \delta\tau'}_{1}$$

$$= \underline{F}(\underline{r}')$$

Returning to our proof of the Helmholtz theorem

$$\nabla^2\left(\tfrac{1}{R}\right) = \underline{\nabla}\cdot\underline{\nabla}\left(\tfrac{1}{R}\right) = -\underline{\nabla}\cdot\left(\hat{R}/R^2\right) = -4\pi\,\delta^3(\hat{\underline{R}})$$

so that

$$\underline{\nabla}\cdot\underline{F} = +\frac{1}{4\pi}\int_{V'} \underbrace{(\underline{\nabla}\cdot\underline{F})'}_{\text{function of the primed co-ordinates}}\, 4\pi\,\delta^3(\hat{\underline{R}})\, d\tau'$$

only contribution to this $\int$ is when $R=0$ $r=r'$

$$\underline{\nabla}\cdot\underline{F} = \underline{\nabla}\cdot\underline{F}$$

QED

<u>Also</u>

$$\begin{aligned}
\underline{\nabla}\wedge\underline{F} &= \underline{\nabla}\wedge\left(-\underline{\nabla}\phi + \underline{\nabla}\wedge\underline{A}\right)\\
&= \underline{\nabla}\wedge\underline{\nabla}\wedge\underline{A}\\
&= -\nabla^2\underline{A} + \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A}\right)
\end{aligned}$$

$$\nabla^2\underline{A} = \frac{1}{4\pi}\int_{V'} \nabla^2\frac{(\underline{\nabla}\wedge\underline{F})'}{|\underline{R}|}\,d\tau' = \int_{V'} \underbrace{(\underline{\nabla}\wedge\underline{F})'}_{\text{primed co-ordinates only}}\,\nabla^2\left(\tfrac{1}{R}\right)\frac{d\tau'}{4\pi}$$

$$\begin{aligned}
&= -\int (\underline{\nabla}\wedge\underline{F})'\,\underline{\nabla}\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right)\frac{d\tau'}{4\pi}\\
&= -\int (\underline{\nabla}\wedge\underline{F})'\,\underbrace{\delta^3(\hat{\underline{R}})}_{\text{zero unless } r=r'}\,d\tau'\\
&= -\underline{\nabla}\wedge\underline{F}
\end{aligned}$$

$$\underline{\nabla}(\underline{\nabla}\cdot\underline{A}) = \frac{\underline{\nabla}}{4\pi}\int_{V'}\underline{\nabla}\cdot\left[\frac{(\underline{\nabla}\wedge\underline{F})'}{|\underline{R}|}\right]d\tau' = \frac{\underline{\nabla}}{4\pi}\left[\int_{V'}(\underline{\nabla}\wedge\underline{F})'\cdot\underline{\nabla}\left(\tfrac{1}{R}\right) + \frac{1}{R}\left[\underline{\nabla}\cdot\underbrace{(\underline{\nabla}\wedge\underline{F})'}_{\text{primed only}}\right]d\tau'\right]$$

$\Rightarrow$ zero

$$= -\frac{\underline{\nabla}}{4\pi}\int_{V'}(\underline{\nabla}\wedge\underline{F})'\cdot\underline{\nabla}'\left(\tfrac{1}{R}\right)d\tau' \qquad \underline{\nabla}\left(\tfrac{1}{R}\right) = -\underline{\nabla}'\left(\tfrac{1}{R}\right)$$

but $\underline{\nabla}' \cdot \left(\frac{(\underline{\nabla} \wedge \underline{F})'}{R} \right) = (\underline{\nabla} \wedge \underline{F})' \cdot \nabla'(1/R) + \underbrace{\frac{\underline{\nabla}' \cdot (\nabla \wedge \underline{F})'}{R}}_{\text{zero}}$ div curl $\underline{A} = 0$ identically

$\Rightarrow$

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = -\frac{\underline{\nabla}}{4\pi} \int_{V'} \underline{\nabla}' \cdot \left[\frac{(\underline{\nabla} \wedge \underline{F})'}{R} \right] d\tau'$$

$$= -\frac{\underline{\nabla}}{4\pi} \oint_{S'} \frac{(\underline{\nabla} \wedge \underline{F})'}{R} \cdot d\underline{a}'$$

but $da' \propto R^2$ @ ∞

i.e. $(\underline{\nabla} \wedge \underline{F})' = 0$ @ $R = \infty$

So long as $(\underline{\nabla} \wedge \underline{F})'$ is zero as we go to infinity then the integral is zero $\Rightarrow$ $\underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = 0$ and

$$\underline{\nabla} \wedge \underline{F} = -\underline{\nabla}^2 A = \underline{\nabla} \wedge \underline{F}$$

Thus we have proved that the "sources" $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \wedge \underline{F}$ are sufficient to define $\underline{F}$ everywhere in space.

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_f}{\varepsilon}$$

$$\underline{\nabla}^2\underline{A} - \mu\varepsilon\frac{\partial^2\underline{A}}{\partial t^2} = -\mu\underline{J}_f$$

and $\underline{\nabla}.\underline{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} = 0$ for the Lorentz condition

with $\Box^2 = \nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}$ — D'Alembertian operator

these become

$$\Box^2\phi = -\rho_f/\varepsilon$$

$$\Box^2\underline{A} = -\mu\underline{J}_f$$

(non-conducting, lin media) of which more later.

* Gauge Transformations

Remember (?) from Chapter 16 that $\underline{A}^+ = \underline{A} + \underline{\nabla}\chi$ works equally well as $\underline{A}$

since $\underline{B} = \underline{\nabla}\wedge\underline{A}$ $\qquad \underline{\nabla}\wedge\underline{\nabla}\chi = 0$ identically

or $\underline{B} = \underline{\nabla}\wedge(\underline{A} + \nabla\chi) = \underline{\nabla}\wedge\underline{A}$

that is we can add to $\underline{A}$ the gradient of a scalar function without changing the physics.

If we allow the possibility of time variation then $\chi = \chi(\underline{r}, t)$, but now

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t}$$

$$= -\underline{\nabla}\phi - \frac{\partial\underline{A}^+}{\partial t} + \underline{\nabla}\left(\frac{\partial\chi}{\partial t}\right)$$

$$\underline{E} = -\underline{\nabla}\underbrace{\left(\phi - \partial\chi/\partial t\right)}_{\phi^+} - \frac{\partial \underline{A}^+}{\partial t}$$

but E must be unchanged

$$\underline{E} = -\underline{\nabla}\phi^+ - \frac{\partial A^+}{\partial t}$$

Thus if we also allow $\phi^+ \rightarrow \phi - \partial\chi/\partial t$ we see that $\underline{E}$ is unchanged.

$$\underline{A}^+ = \underline{A} + \underline{\nabla}\chi \qquad \phi^+ = \phi - \partial\chi/\partial t$$

are called gauge transformations. Maxwell's equations are invariant to a gauge transformation.

If we demand that $\underline{A}$ and ϕ satisfy the Lorentz condition then

$$\underline{\nabla}\cdot\underline{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} = 0$$

$$\Rightarrow \underline{\nabla}\cdot\underline{A}^+ - \underline{\nabla}\cdot\underline{\nabla}\chi + \mu\varepsilon\frac{\partial\phi^+}{\partial t} + \mu\varepsilon\,\partial^2\chi/\partial t^2 = 0$$

$$\underline{\nabla}\cdot\underline{A}^+ + \mu\varepsilon\frac{\partial\phi^+}{\partial t} = \nabla^2\chi - \mu\varepsilon\frac{\partial^2\chi}{\partial t^2}$$

For $\underline{A}^+$ and ϕ^+ to satisfy the Lorentz condition also

$$\nabla^2\chi - \mu\varepsilon\frac{\partial^2\chi}{\partial t^2} = \Box^2\chi = 0$$

(we can also easily verify that $\Box^2\phi^+ = -\rho_f/\varepsilon$ and $\Box^2\underline{A}^+ = -\mu J_f$ - leaving "the physics" unaltered)

Coulomb
e.g. Lorentz

Remember — a different condition on $\underline{A}$ and ϕ will lead to different requirements on χ

* In particular in chapter 16 when we first introduced the vector potential we demanded that

$$\underline{\nabla} \cdot \underline{A} = 0 \quad \text{(equivalent to Lorentz condition)}$$

and with this condition $\nabla^2 \chi = 0$

This situation is known as the Coulomb gauge and arises naturally when there is no time dependence. (in a non-conducting medium, $\sigma = 0$)

Chapter 22 — Problems : 2, 5, 7, 8

* Note that the gauge transformation invariance is basic. The scalar function χ is arbitrary until we define $\underline{\nabla} \cdot \underline{A}$. Then demanding that $\underline{\nabla} \cdot \underline{A}$ and $\underline{\nabla} \cdot \underline{A}'$ behave in the same way places conditions on χ.

$\underline{\nabla} \cdot \underline{A}$ can be chosen in any manner which will result in a condition on χ which does not involve ϕ

e.g. $\underline{\nabla} \cdot \underline{A} = + \phi^2$ will not work