24 Plane Waves

* In terms of $\underline{E}$ and $\underline{B}$ Maxwell's equations are given by

eqns (21.42) – (21.4[illegible])

$$\left\{ \begin{array}{ll} \underline{\nabla} \cdot \underline{E} = \rho_f / \varepsilon & \underline{\nabla} \wedge \underline{E} = -\partial \underline{B}/\partial t \\ \underline{\nabla} \cdot \underline{B} = 0 & \underline{\nabla} \wedge \underline{B} = \mu\sigma \underline{E} + \mu \underline{J}_f' + \mu\varepsilon\, \partial \underline{E}/\partial t \end{array} \right.$$

for l.i.h media. [Unless specified we assume l.i.h media from now on]

If $\rho_f = \underline{J}_f' = 0$ then

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \wedge \underline{E} = -\partial \underline{B}/\partial t$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \wedge \underline{B} = \mu\sigma \underline{E} + \mu\varepsilon\, \partial \underline{E}/\partial t$$

taking the curl of the last equation

$$\underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{B}) = -\underline{\nabla}^2 \underline{B} + \underline{\nabla}(\underline{\nabla} \cdot \underline{B}) = \mu\sigma \underline{\nabla} \wedge \underline{E} + \mu\varepsilon \frac{\partial}{\partial t}(\underline{\nabla} \wedge \underline{E})$$

but $\underline{\nabla} \wedge \underline{E} = -\dfrac{\partial \underline{B}}{\partial t}$ and $\underline{\nabla} \cdot \underline{B} = 0$

$$\Rightarrow \quad -\underline{\nabla}^2 \underline{B} = -\mu\sigma\, \partial \underline{B}/\partial t - \mu\varepsilon\, \partial^2 \underline{B}/\partial t^2$$

$$\underline{\nabla}^2 \underline{B} - \mu\sigma \frac{\partial \underline{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{B}}{\partial t^2} = 0$$

and similarly $\underline{\nabla}^2 \underline{E} - \mu\sigma \dfrac{\partial \underline{E}}{\partial t} - \mu\varepsilon \dfrac{\partial^2 \underline{E}}{\partial t^2} = 0$

We have separated the equations and have found that $\underline{E}$ and $\underline{B}$ are solutions of the same equation.

* If we now restrict ourselves to a <u>non-conducting</u> medium, $\sigma = 0$ and

$$\underline{\nabla}^2 \underline{E} = \mu\varepsilon \, \partial^2 E/\partial t^2 \qquad \text{(similarly for } \underline{B}\text{)}$$

for each co-ordinate of $\underline{E}$ ($\underline{B}$) we obtain for example

$$\underbrace{\nabla^2 E_x = \mu\varepsilon \, \partial^2 E_x/\partial t^2}_{\text{3 dimensional wave equation}}$$

* For <u>plane waves</u> propagating in the z-direction we have

$$E_x = E_x(z,t) \; ; \; E_y = E_y(z,t) \; ; \; E_z = E_z(z,t)$$

i.e. no dependence on x and y.

$\Rightarrow$

$$\underbrace{\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}}_{\text{1 dimensional wave equation}} \; ; \; \frac{\partial^2 E_y}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} \; ; \; \frac{\partial^2 E_z}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_z}{\partial t^2}$$

Thus all the components of $\underline{E}$ ($\underline{B}$) satisfy the 1-D wave equation.

A solution of one component will therefore give us all the other solutions

* For $E_x(z,t)$ try $E_x = Z(z)T(t)$ i.e. separate the variables

then
$$T \frac{\partial^2 Z}{\partial z^2} = \mu\varepsilon Z \frac{\partial^2 T}{\partial t^2}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{\mu\varepsilon}{T}\frac{\partial^2 T}{\partial t^2} = \text{constant} = \overbrace{-}^{\text{Chosen negative because + leads to exponentially decaying solutions – not interesting}}k^2$$

then
$$\frac{\partial^2 Z}{\partial z^2} + Zk^2 = 0 = \frac{\partial^2 T}{\partial t^2} + \frac{Tk^2}{\mu\varepsilon}$$

ω – angular frequency $\qquad (k^2/\mu\varepsilon) = \omega^2$

General solutions may then be written as

$$Z_k = \alpha_k e^{ikz} + \beta_k e^{-ikz} \quad ; \quad T = \gamma_k e^{i\omega t} + \delta_k e^{-i\omega t}$$

where α_k β_k γ_k δ_k are constants (which depend on the value of k)

The general solution is a sum over all values of k

$$E_x = \sum_k \left(\alpha_k e^{ikz} + \beta_k e^{-ikz}\right)\left(\gamma_k e^{i\omega t} + \delta_k e^{-i\omega t}\right)$$

$$= \sum_k \left(\alpha_k \delta_k e^{i(kz-\omega t)} + \beta_k \gamma_k e^{-i(kz-\omega t)}\right)$$

$$+ \sum_k \left(\alpha_k \gamma_k e^{i(kz+\omega t)} + \beta_k \delta_k e^{-i(kz+\omega t)}\right)$$

Now we assume that $\omega > 0$ but k can be positive or negative. Each of the above terms represents a plane wave moving in $+z$ or $-z$ direction with velocity

$$v = \omega/k = \frac{k}{k\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

If $k > 0$ then $v > 0$, but if $k < 0$ then $v < 0$.

Considering $\underbrace{e^{i(kz-\omega t)}}_{\substack{+z \text{ direction} \\ v>0}}$; if $k < 0$ it becomes $\underbrace{e^{-i(|k|z+\omega t)}}_{-z \text{ direction } v<0}$

Thus by considering $e^{i(kz-\omega t)}$ and allowing $k > 0$ and $k < 0$ we include waves travelling in $+z$ and $-z$ direction.

The other terms in the summations are complex conjugates of the above two waves. Therefore it is sufficient to consider a wave of the form $e^{i(kz-\omega t)}$

For physical solutions we will be considering the real part.

* In particular we will consider a single value of k then

$$E_x = E_{0x} e^{i(kz-\omega t)}$$

E_{0x} is amplitude, which may be complex

$$E_{0x} = \underbrace{{}_aE_{0x}}_{\text{real}}\, e^{i\theta}$$ phase angle.

so that $E_x = {}_aE_{0x}\, e^{i(kz-\omega t+\theta)}$

But $\underline{E}_x$ must be real — $\underline{E}_x$ is a physical quantity.
Thus

$$E_x^{\text{physical}} = {}_aE_{0x} \cos(kz-\omega t+\theta)$$

(real part of the solution)

(If $\theta = 0$ there is no phase factor θ in the real amplitude)

* All the components of $\underline{E}$ and $\underline{B}$ will have the same form since they all satisfied the same equation
$\Rightarrow$

$$\underline{E} = \underline{E}_0\, e^{i(kz-\omega t)} \quad ; \quad \underline{B} = \underline{B}_0\, e^{i(kz-\omega t)} \qquad (*)$$

for a plane wave in z direction

$$\underline{\nabla}\cdot\underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = ikE_{z0}\, e^{i(kz-\omega t)} = ikE_z$$

$$\underline{\nabla}\cdot\underline{B} = ikB_z$$

and

$$\underline{\nabla}\wedge\underline{E} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\hat{\underline{x}}\, ikE_{y0}\, e^{i(kz-\omega t)} + \hat{\underline{y}}\, ikE_{x0}\, e^{i(kz-\omega t)}$$

$$= ik\left[\hat{\underline{y}} E_x - \hat{\underline{x}} E_y\right] = ik\,(\hat{\underline{z}}\wedge \underline{E})$$

$$\underline{\nabla}\wedge\underline{B} = ik\left[\hat{\underline{y}} B_x - \hat{\underline{x}} B_y\right] = ik\,(\hat{\underline{z}}\wedge \underline{B})$$

Substituting these expressions into Maxwells equations we find

$$\underline{\nabla}\cdot\underline{E} = 0 = ikE_z \Rightarrow k\,\hat{\underline{z}}\cdot\underline{E} = 0$$

$$\underline{\nabla}\cdot\underline{B} = 0 = ikB_z \Rightarrow k\,\hat{\underline{z}}\cdot\underline{B} = 0$$

$$\underline{\nabla}\wedge\underline{E} = ik\left[\hat{\underline{y}} E_x - \hat{\underline{x}} E_y\right] = ik(\hat{\underline{z}}\wedge\underline{E}) = -\partial B/\partial t = +i\omega\underline{B}$$

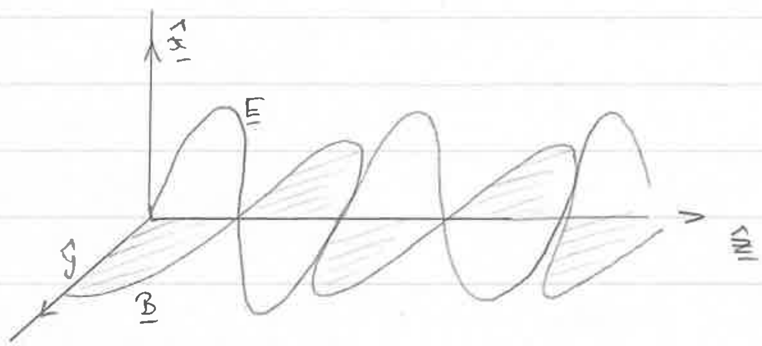
$$\underline{\nabla}\wedge\underline{B} = ik\left[\hat{\underline{y}} B_x - \hat{\underline{x}} B_y\right] = ik(\hat{\underline{z}}\wedge\underline{B}) = \mu\varepsilon\,\partial E/\partial t = -i\mu\varepsilon\omega\underline{E}$$

For a non-static case — $k \neq 0$, $\omega \neq 0$ — we have both E_z, B_z zero $\Rightarrow$ the wave must be transverse and since $ik(\hat{\underline{z}} \wedge \underline{E}) = i\omega \underline{B}$

$ik(\hat{\underline{z}} \wedge \underline{B}) = -\dfrac{\omega i}{v^2}\underline{E}$

Substituting exponential forms of $\underline{E}$ + $\underline{B}$ (*), then writing $\underline{E}_0 = E_0 e^{i\Theta}$ leads us to the conclusion that $\underline{B}_0 = B_0 e^{i\Theta}$ – with same phase angle Θ.

$\underline{E}$ and $\underline{B}$ are mutually perpendicular and both perpendicular to $\hat{\underline{z}}$. $\underline{E}$ and $\underline{B}$ are in the xy plane.
The form of the wave is then indicated below



N.B. I have drawn $\underline{E}$ along x and $\underline{B}$ along y – this is for convenience. The only restriction is that $\underline{E}$ and $\underline{B}$ are perpendicular. Also only linearly polarised $\underline{E}, \underline{B}$ are shown. Real EM radiation will have $\underline{E}, \underline{B}$ at all directions in xy plane.

* Notice that $\underline{E}$ and $\underline{B}$ are in phase, they both have the same phase angle (see top of page)

$$\underline{E}_{real} = \underline{E}_{0a}\cos(kz - \omega t + \Theta)\ ;\ \underline{B}_{real} = \underline{B}_{0a}\cos(kz - \omega t + \Theta)$$

Also note that the Poynting vector $S = (\underline{E} \wedge \underline{B})/\mu$

$$S = \frac{(\underline{E} \wedge \underline{B})}{\mu} = \underline{E} \wedge (\hat{\underline{z}} \wedge \underline{E})\frac{k}{\omega\mu}$$

$$= \hat{\underline{z}}\frac{E^2 k}{\omega\mu} = \hat{\underline{z}}\frac{E^2 \cancel{k}\sqrt{\mu\varepsilon}}{\cancel{k}\,\mu} = \hat{\underline{z}} E^2 \sqrt{\frac{\varepsilon}{\mu}}$$

$\omega = \left(\dfrac{k}{\sqrt{\mu\varepsilon}}\right)$

To summarise (for now)

* For plane waves : $\underline{E} = \underline{E}_0 e^{i(kz-\omega t)}$

and we associate all em waves with this type of solution to Maxwells equations

ω — angular frequency

k — propagation constant (wave number)

$(\omega = 2\pi\nu)$ ν — frequency

$T = 1/\nu$ — period

$\lambda = 2\pi/k$ — wavelength $\left[\lambda = v/\nu = \frac{\omega}{k}\frac{2\pi}{\omega} = \frac{2\pi}{k}\right]$

$v = \omega/k = \nu\lambda$ — phase velocity

$(kz - \omega t)$ — phase

Θ — phase factor

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

* In vacuo $v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ — Speed of light (" " all em radiation)

* Since $i\omega\underline{B} = ik(\hat{z}\wedge\underline{E})$

$$|\underline{B}| = \frac{k}{\omega}|\underline{E}| = \frac{\cancel{k}(\mu\varepsilon)^{1/2}}{\cancel{k}}|\underline{E}| = \frac{|\underline{E}|}{v}$$

in vacuo $|\underline{B}| = |\underline{E}|/c$ [magnitude of $\underline{B}$ field is much smaller than $\underline{E}$]

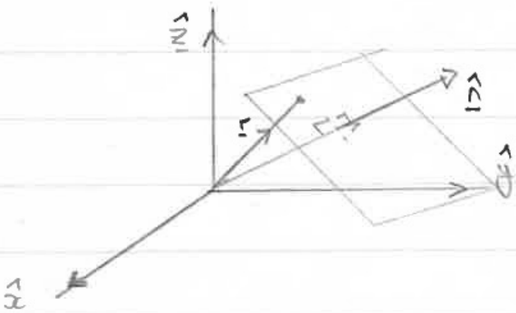
* n = refractive index $= c/v$ (≥ 1)

$$n = \frac{\sqrt{\mu_0\varepsilon_0 K_\mu K_e}}{\sqrt{\mu_0\varepsilon_0}} = \sqrt{K_e K_\mu}$$

(Sect 24.5)

* Plane Waves In an Arbitrary Direction

So far, for simplicity we have assumed that our plane wave traveled in the z-direction. In other words, we defined the z-axis as the direction of propagation. Assume now that the axes are fixed and the wave travels in some arbitrary direction ($\hat{n}$)



$\underline{r}$ is the position vector of a point on the plane

$\hat{\underline{n}}$ is normal to the plane

Then $\zeta = \hat{\underline{n}} \cdot \underline{r}$ = distance of plane from origin.

ζ will now take the place of z in our plane wave expressions

$$\text{e.g.} \quad E_x = E_{0x} e^{i(k\zeta - \omega t)} = E_{0x} e^{i(k\hat{\underline{n}}\cdot\underline{r} - \omega t)}$$

If we define $\underline{k} = k\hat{\underline{n}}$ propagation vector then

$$E_x = E_{0x} e^{i(\underline{k}\cdot\underline{r} - \omega t)} = E_{0x} e^{i(k_x x + k_y y + k_z z - \omega t)}$$

with similar expressions for E_y and E_z

$$\frac{\partial E_x}{\partial x} = i k_x E_x \, ; \quad \frac{\partial E_y}{\partial y} = i k_y E_y \, ; \quad \frac{\partial E_z}{\partial z} = i k_z E_z \, ;$$

$$\underline{\nabla} \cdot \underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \underline{i\,\underline{k} \cdot \underline{E}} = 0$$

Similarly $\underline{\nabla} \cdot \underline{B} = 0 \implies \underline{i\,\underline{k} \cdot \underline{B}} = 0$

$$\underline{\nabla} \wedge \underline{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\underline{\nabla} \wedge \underline{E} = \left[\hat{\underline{x}}\left(k_y E_z - k_z E_y\right) + \hat{\underline{y}}\left(k_z E_x - k_x E_z\right) + \hat{\underline{z}}\left(k_x E_y - k_y E_x\right)\right] i$$

$$= \left(\underline{k} \wedge \underline{E}\right) i \;\; = -\frac{\partial \underline{B}}{\partial t} \;=\; + i\omega \underline{B}$$

or $\underline{\left(\underline{k} \wedge \underline{E}\right) = \omega \underline{B}}$

Similarly we can obtain $\underline{\underline{k} \wedge \underline{B} = -\mu\varepsilon\omega \underline{E}}$

These four relationships imply that, as in the case of propagation along the z-axis, $\underline{E}$ and $\underline{B}$ are perpendicular to $\underline{k}$ and each other. A transverse wave. And furthermore that $\underline{E}$ and $\underline{B}$ are <u>in phase</u>

$$\underline{B} = \frac{1}{\omega}\left(\underline{k} \wedge \underline{E}\right) = \frac{k}{\omega}\left(\hat{\underline{k}} \wedge \underline{E}\right)$$

but $V = \omega/k = 1/\sqrt{\mu\varepsilon}$

$$\Rightarrow \quad \underline{B} = \left(\hat{\underline{k}} \wedge \underline{E}\right)\sqrt{\mu\varepsilon} = \frac{\left(\hat{\underline{k}} \wedge \underline{E}\right)}{Z}\mu \quad \left(= \mu \underline{H}\right)$$

where $Z = \left(\mu/\varepsilon\right)^{1/2}$ <u>Wave impedance</u>

Impedance of the vacuum $Z_0 = \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} = \left(\dfrac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}\right)^{1/2}$

$\simeq 377$ ohms

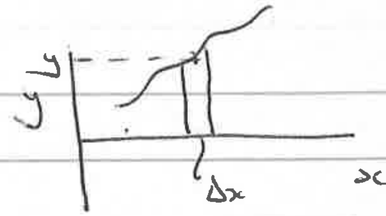
[Recall in ac circuits $Z = V/I$; here we associate V with E and I with H; $Z = |\underline{E}|/|\underline{H}|$]

cf $\underline{S} = \hat{z} E^2\sqrt{\varepsilon/\mu} = \hat{z} E^2/Z$

* <u>Energy Flow in Waves</u>

Remember our solutions of Maxwell's equations

For discrete variable



$$\langle y \rangle = \frac{\sum_i y\, \Delta x}{\sum_i \Delta x}$$

Where summation is over range for which $\langle y \rangle$ is to be calculated.

For continuous variable

$$\langle y \rangle = \int y\, dx \Big/ \int dx$$

When $y = \cos^2 x$ $\quad \langle y \rangle = \int_0^{\pi} \cos^2 x\, dx \Big/ \int_0^{\pi} dx$

$$= \frac{\pi/2}{\pi} = 1/2$$

are written for example

$$E_x = E_{0x} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

where E_{0x} can be complex

Similarly we can write in vector form

$$\underline{E} = \underline{E}_0 \, e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

where $\underline{E}_0$ can be a complex vector.

As far as energy is concerned we will generally, practically, be interested in time averages, since $\omega(\nu)$ is often too large for our detection equipment to follow the variation with time. Therefore we will absorb the spacial variation of $\underline{E}$ into $\underline{E}_0$ leaving (equivalent to observing at a particular position)

$$\underline{E} = \underline{E}_0 e^{-i\omega t} \quad \text{and} \quad \underline{B} = \underline{B}_0 e^{-i\omega t}$$

With $\underline{E}_0 = \underline{E}_R + i\underline{E}_I \qquad \underline{B}_0 = \underline{B}_R + i\underline{B}_I$

$$\Rightarrow \underline{E} = (\underline{E}_R + i\underline{E}_I)(\cos\omega t - i\sin\omega t)$$

$$\underline{E} = \underline{E}_R \cos\omega t + \underline{E}_I \sin\omega t + i(\underline{E}_I \cos\omega t - \underline{E}_R \sin\omega t)$$

The real part of $\underline{E}$ is the physically observable quantity

$\Rightarrow$

$$\underline{E}_{Real} = \underline{E}_R \cos\omega t + \underline{E}_I \sin\omega t$$

$$\text{and } \underline{B}_{Real} = \underline{B}_R \cos\omega t + \underline{B}_I \sin\omega t$$

these are physical E, B, etc. $\Rightarrow$

Energy flow is defined by the Poynting vector $\underline{S} = (\underline{E}\wedge\underline{B})/\mu$ $= \underline{E}\wedge\underline{H}$

thus

$$\underline{S} = \frac{1}{\mu}\Big[(\underline{E}_R\wedge\underline{B}_R)\cos^2\omega t + (\underline{E}_I\wedge\underline{B}_I)\sin^2\omega t + [(\underline{E}_R\wedge\underline{B}_I) + (\underline{E}_I\wedge\underline{B}_R)]\sin\omega t\cos\omega t\Big]$$

Taking the average over time, using $\langle\cos^2\omega t\rangle = \frac{1}{2} = \langle\sin^2\omega t\rangle$ and $\langle\sin\omega t\cos\omega t\rangle = 0$

$$\langle\underline{S}\rangle = \frac{1}{2\mu}\left[(\underline{E}_R\wedge\underline{B}_R) + (\underline{E}_I\wedge\underline{B}_I)\right]$$

(p 392)

It can be shown that $\langle \underline{S} \rangle$ can also be written in terms of the complex forms of $\underline{E}$ and/or $\underline{B}$ $\underbrace{(\underline{E}_c, \underline{B}_c)}_{\text{includes real + imag part.}}$

$$e^{-i\omega t}\langle \underline{S} \rangle_{e^{i\omega t}} = \tfrac{1}{2\mu}\,\mathrm{Real}\left(\underline{E}_c \wedge \underbrace{\underline{B}_c^*}_{\text{complex conjugate } i \to -i \text{ in } \underline{B}_c}\right)$$

$$(\underline{E}_c \wedge \underline{B}_c^*) = (\underline{E}_R + i\underline{E}_I) \wedge (\underline{B}_R - i\underline{B}_I) = [\underline{E}_R \wedge \underline{B}_R + \underline{E}_I \wedge \underline{B}_I] + i[\underline{E}_I \wedge \underline{B}_R - \underline{E}_R \wedge \underline{B}_I]$$

For a plane wave $\underline{B}_c = (\underline{k} \wedge \underline{E}_c)/\omega$

and

$$\langle \underline{S} \rangle = \tfrac{1}{2\mu}\,\mathrm{Real}\left[\underline{E}_c \wedge (\underline{k} \wedge \underline{E}_c^*)\right]/\omega$$

additional factor of 1/2 relative to non-time-averaged value of $\underline{S}$

$$= \hat{\underline{k}}\,\frac{k}{2\mu\omega}(E_c E_c^*) = \hat{\underline{k}}\,\frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2} E_0^2$$

$$= \hat{\underline{k}}\,\frac{1}{2}\left(\frac{\mu}{\varepsilon}\right)^{1/2} H_0^2$$

$E_0 = H_0 \underbrace{\left(\frac{\mu}{\varepsilon}\right)^{1/2}}_{Z}$

* The energy density $u_e = \tfrac{1}{2}\varepsilon E^2$ (this is $\underline{E}$ "physical")

$$= \tfrac{1}{2}\varepsilon\left(E_R \cos\omega t + E_I \sin\omega t\right)^2$$

$$= \tfrac{1}{2}\varepsilon\left[E_R^2 \cos^2\omega t + E_I^2 \sin^2\omega t + 2E_R E_I \cos\omega t \sin\omega t\right]$$

Averaging over time $\langle u_e \rangle = \tfrac{1}{4}\varepsilon\left(E_R^2 + E_I^2\right)$

$$= \tfrac{1}{4}\varepsilon\,\underline{E}_0^2$$

Using the fact that $\underline{B}_0 = (\underline{k} \wedge \underline{E}_0)/\omega$ and $\underline{H} = \underline{B}/\mu$ we obtain that $\langle u_e \rangle = \tfrac{1}{4}\varepsilon E_0^2 = \tfrac{1}{4}\mu H_0^2 = \langle u_m \rangle$

thus

$$\underbrace{\langle u \rangle}_{\text{time averaged energy density of complete wave.}} = \langle u_e \rangle + \langle u_m \rangle = \underbrace{\tfrac{1}{2}\varepsilon E_0^2 = \tfrac{1}{2}\mu H_0^2}_{\text{as expected}}$$

<u>OR</u>

$$\langle \underline{S} \rangle = \hat{\underline{k}}\,\frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2} E_0^2 = \frac{\hat{\underline{k}}\,\langle u \rangle}{\sqrt{\mu\varepsilon}} = \hat{\underline{k}}\,v\,\langle u \rangle$$

$$\underline{\langle \underline{S} \rangle = \underline{V}\,\langle u \rangle}$$

Notice the similarity between $\langle \underline{S} \rangle = \underline{v} \langle u \rangle$
and $\underline{J} = \underline{v}\, \rho$
$\underline{K} = \underline{v}\, \sigma$

$\langle \underline{S} \rangle$ – energy "current", $\langle u \rangle$ – "current" density
(energy)

* So far we have seen that in a l.i.h medium, where $\sigma, \rho_f, \underline{J}_f$ are all zero we can associate the solutions of Maxwell's equations with transverse waves, in which $\underline{E}$ and $\underline{B}$ oscillate, in phase, at right angles. These waves are interpreted as the well known EM spectrum.

As you know elements of the EM spectrum can be polarised. How is this polarisation described in terms of the $\underline{E}$ and $\underline{B}$ ($\underline{H}$) vectors?

* Assuming propagation in the z-direction we may write the $\underline{E}$ part of the wave as

$$\underline{E} = \big(\underbrace{E_1 e^{i\Theta_1}}\,\hat{\underline{x}} + \underbrace{E_2 e^{i\Theta_2}}\,\hat{\underline{y}}\big)\, e^{i(kz - \omega t)}$$

Complex amplitudes in $\hat{\underline{x}}$ and $\hat{\underline{y}}$ directions.

$$\Rightarrow \quad E_x = E_1 \cos(kz - \omega t + \Theta_1)$$
$$E_y = E_2 \cos(kz - \omega t + \Theta_2)$$

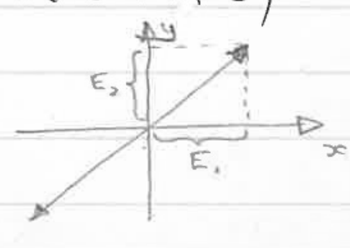
(E_1, E_2 are now real and represent the max $\underline{E}$ in the $\hat{\underline{x}}$ and $\hat{\underline{y}}$ directions)

There are several interesting cases to consider

1) $\Theta_1 = \Theta_2 = \Theta$

$$E_x = E_1 \cos(kz - \omega t + \Theta)$$
$$E_y = E_2 \cos(kz - \omega t + \Theta)$$

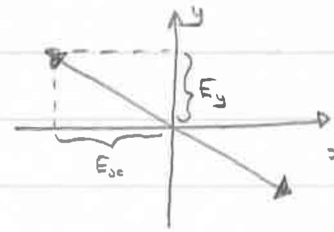
E_x, E_y are in phase
<u>Linearly polarised</u>



2) $\Theta_1 = \pi + \Theta_2$

$$E_x = E_1 \cos(kz - \omega t + \pi + \Theta_2)$$
$$= -E_1 \cos(kz - \omega t + \Theta_2)$$
$$E_y = E_2 \cos(kz - \omega t + \Theta_2)$$

E_x, E_y out of phase by π



<u>Linearly polarised</u>

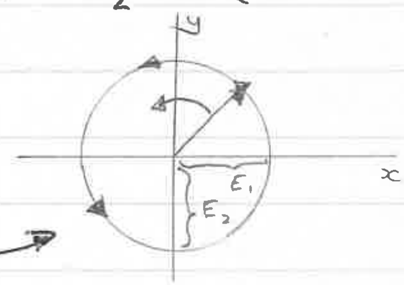
3) $\Theta_1 = \pi/2 + \Theta_2$

E_x, E_y out of phase by $\pi/2$

$$E_x = E_1 \cos(\pi/2 + kz - \omega t + \Theta_2)$$
$$E_x = -E_1 \sin(kz - \omega t + \Theta_2)$$
$$E_y = E_2 \cos(kz - \omega t + \Theta_2)$$

$$\underbrace{\frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_2^2} = 1}_{\text{Ellipse}}$$

When $E_1 = E_2 \Rightarrow$ circle



<u>Circularly polarised</u>

4) For other relative values of Θ_1 and Θ_2 we obtain <u>elliptically</u> polarised $\underline{E}$ fields, with major and minor axes, not necessarily along $\hat{\underline{x}}$ and $\hat{\underline{y}}$.

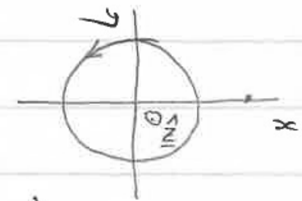
* In the case of circularly (or elliptically) polarised $\underline{E}$ fields it is possible for the $\underline{E}$ vector to rotate in either direction (clockwise or anticlockwise). The direction of rotation is defined by the sign of $\Theta_1 - \Theta_2$.

Using $P = kz - \omega t + \Theta_1$ and $\Delta = \Theta_1 - \Theta_2$

$$E_x = E_1 \cos P$$
$$E_y = E_1 \cos(P - \Delta)$$

For <u>P increasing</u>

When $\Delta > 0$ $(\theta_1 > \theta_2)$ then E_x leads E_y and we have anti-clockwise rotation of the $\underline{E}$ vector. (Note that we will always assume that the direction of propagation is out-of the page)



Of course for $\Delta < 0$ $(\theta_1 < \theta_2)$ the $\underline{E}$ vector rotates in a clockwise manner. (E_y leads E_x)

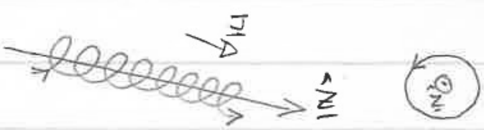
Another way of looking at this is as follows:

P depends on both z and t.

a) Fixed t, z variable (snapshot of wave)

As z increases P increases and when $\Delta > 0$ we have anticlockwise rotation

positive helicity $\Delta > 0$

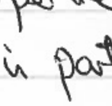


IF $\Delta < 0$ negative helicity

b) Fixed z, t variable

As t increases P decreases and our conclusions are all reversed.

$\Delta < 0$ right handed polarisation ($\equiv$ pos helicity)



$\Delta > 0$ left handed polarisation ($\equiv$ neg helicity)

[Helicity, and right handed / left handedness are properties which are particularly useful in handling spins in particle physics]

Here it may be advantageous to move on to the discussion of reflection / refraction in Chapter 25 – returning finally to the topic of conducting media. GO TO page 203 in these notes.

Problems from 24 without conductivity 24-3, 5, 9, 13, 15, 17, 19, 31