* We now turn to the situation where $\sigma \neq 0$ i.e. the conducting medium. But ρ_f and $\underline{J}_f$ still remain zero.

With $\sigma \neq 0$ we were able to reduce Maxwell's equations to (p375 - Chap 24)

$$\underline{\nabla}^2 \underline{E} - \mu\sigma\, \partial \underline{E}/\partial t - \mu\varepsilon\, \partial^2 \underline{E}/\partial t^2 = 0$$
$$\underline{\nabla}^2 \underline{B} - \mu\sigma\, \partial \underline{B}/\partial t - \mu\varepsilon\, \partial^2 \underline{B}/\partial t^2 = 0$$

$\underline{E}$, $\underline{B}$ still satisfy the same equation, the difference from the non-conducting case being that we now have a damping term $[-\mu\sigma\, \partial \underline{E}/\partial t]$

Once again, for simplicity, we assume a plane wave $E_x(z,t)$ traveling in z then, - only z dependence of fields

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\sigma \frac{\partial E_x}{\partial t} - \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

with similar equations for E_y E_z B_x B_y B_z.

Attempting a solution of the form $E_x = E_{0x} e^{i(kz-\omega t)}$

$$-k^2 E_x + i\mu\sigma\omega E_x + \mu\varepsilon\omega^2 E_x = 0$$
$$k^2 = \mu\varepsilon\omega^2 + i\mu\omega\sigma \quad (\text{cf } k = \omega/v = \omega(\mu\varepsilon)^{1/2} \text{ non-conducting})$$

That is the propagation constant k is a complex quantity. k can be represented either as $\alpha + i\beta$ or $|k|e^{i\phi}$

After some algebra we obtain

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[\left[1 + \left(\frac{1}{Q^2} \right) \right]^{1/2} + 1 \right]^{1/2} \qquad Q = \frac{\omega\varepsilon}{\sigma}$$

$$\beta = \omega \left(\frac{\mu\varepsilon}{2} \right)^{1/2} \left[\left(1 + \frac{1}{Q^2} \right)^{1/2} - 1 \right]^{1/2}$$

[When $\sigma = 0$ $(Q = \infty)$ $\beta = 0$ and $\alpha = \omega/v = k$ as expected]

Note on interpretation of Q

$$Q = \frac{\omega \varepsilon}{\sigma}$$

$J_f = \sigma E$

Remember Maxwell's equation $\nabla \wedge \underline{B} = \underbrace{\mu J_f}_{\text{Conduction Current}} + \underbrace{\mu \varepsilon \partial E/\partial t}_{\text{Displacement Current.}}$

If $E = E(\underline{r}) e^{-i\omega t}$

(sinusoidal wave)

then $|\partial E/\partial t| = E\omega$

$$\Rightarrow \frac{|\text{Displacement current}|}{|\text{Conduction current}|} = \frac{\mu \varepsilon \, \partial E/\partial t}{\mu \sigma E} = \frac{\cancel{\mu} \varepsilon \omega \cancel{E}}{\cancel{\mu} \sigma \cancel{E}} = Q$$

thus Q is a measure of the relative importance of the conduction and displacement currents in the medium.

For $Q \gg 1$ (σ small, or ε, ω v. large)
(usually insulator)
Conduction current is less important than displacement current

$Q \ll 1$ (σ large, ω, ε – small)
(usually conductor)
Conduction current more important than displacement current.

N.B. It is important to realise that Q is frequency dependent eg. at high enough frequency (large ω) a conductor will begin to take on the properties of an insulator. Although in practise, quantum effects associated with the discrete nature of photons become important before this takes place.

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and $|k| = (\alpha^2+\beta^2)^{1/2} = \omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{Q^2}\right]^{1/4}$

$\tan\Omega = \beta/\alpha = (1+Q^2)^{1/2} - Q$

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* But with a complex k

$$E_x = \underbrace{E_{0x}e^{-\beta z}}_{\text{amplitude damped by } e^{-\beta z}} e^{i(\alpha z - \omega t)}$$

Note that if $\beta = 0$, when $\sigma = 0$ there is no damping. Energy of the wave is lost due to resistive heating in the conductor.

1) Velocity of wave $v = (\omega/\alpha)$

$$= \underbrace{\frac{1}{\sqrt{\mu\varepsilon}}}_{\text{speed in a nonconductor with same } \mu \text{ and } \varepsilon} \underbrace{\left[\frac{2}{[1+(1/Q^2)]^{1/2}+1}\right]^{1/2}}_{\leq 1}$$

$Q \ll 1$ conductors

Thus the wave will move more slowly in the conducting medium compared to the equivalent non-conducting medium. Furthermore, since Q depends on ω, v depends on ω — the medium is dispersive: waves of different frequency will move at different velocity. [cf white light traveling through glass is split into its constituents forming the spectrum]

Of course when σ is small (Q large) v is almost the same as in the equivalent non-conductor $v = 1/\sqrt{\mu\varepsilon}$.

An alternative interpretation of velocity would be to define

$$V = \frac{\omega}{k} = \frac{\omega}{\alpha+i\beta} = \frac{\omega e^{-i\Omega}}{|k|}$$

then $N = \frac{c}{V} = \frac{ck}{\omega} = \frac{c}{\omega}(\alpha + i\beta) = \frac{c}{v} + i\left(\frac{c\beta}{\omega}\right)$

$$N = n' + i\left(\frac{c\beta}{\omega}\right)$$

Complex index of refraction — actual/normal index of refraction — Imaginary part associated with attenuation.

Sometimes useful notation

2) Attenuation distance or skin depth

The damping factor, $e^{-\beta z}$, in the wave amplitude leads to an attenuation distance or skin depth defined by

$$\delta = 1/\beta = \frac{1}{\omega}\left(\frac{2}{\mu\varepsilon}\right)^{1/2}\left[\left(1 + \frac{1}{Q^2}\right)^{1/2} - 1\right]^{-1/2} \qquad Q = \frac{\omega\varepsilon}{\sigma}$$

If $Q \ll 1$ (σ large – good conductor e.g. metals)
then

$$\delta \simeq \left(\frac{2}{\mu\sigma\omega}\right)^{1/2} \qquad \text{and } v \simeq \left(\frac{2\omega}{\mu\sigma}\right)^{1/2}$$

$$\delta \simeq \frac{v}{\omega} = \frac{\lambda}{2\pi} \qquad [v = \omega/k,\ k = 2\pi/\lambda]$$

where λ is the wavelength of the wave.

e.g. For Cu with $\sigma = 6\times10^7\ (\text{ohm.m})^{-1}$

$$\mu \simeq \mu_0 = 4\pi\times10^{-7}$$
$$\omega = 2\pi\nu$$

then $\delta \simeq (6.5\times10^{-2})\,\nu^{-1/2}$

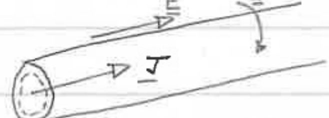
the higher the frequency the smaller δ

e.g. green light $\nu = 6\times10^{14}$ Hz

$$\delta \simeq 2.6\times10^{-9}\ \text{m}$$

The Skin Effect (not in Wangsness text)

When a current flows along a wire it creates an EM field outside (and inside) the wire. The Poynting theorem tells us that in the steady state situation the energy dissipated as heat is exactly equal to the energy passing into the conductor from the EM field. That is an EM field is passing into the conductor.



For time varying currents (a.c.) this is equivalent to the passage of an EM wave, of the same frequency as the ac, into the conductor. But we have just shown that in such a case the EM field is restricted to the skin-depth (δ). Thus the current flows only in a thin region close to the surface.
$\Rightarrow$ For high frequency current transmission a thin tube is as good a conductor as solid rods. [@ 1000 Hz, $\delta \sim 2$ mm]

* Just as in the non-conducting case substitution of the wave form $E_x = E_{0x} e^{-\beta z} e^{i(\alpha z - \omega t)}$ $= E_{0x} e^{i(kz - \omega t)}$ $(k = \alpha + i\beta)$ into Maxwells equations yields

$\nabla \wedge B = \mu\sigma E + \mu\epsilon \frac{\partial E}{\partial t}$

$$k(\hat{z} \cdot E) = 0 = k(\hat{z} \cdot B)$$

$$k(\hat{z} \wedge E) = \omega B \qquad k(\hat{z} \wedge B) = -(\mu\epsilon\omega + i\mu\sigma) E$$

thus E, B are still @ $90°$ to each other and $\hat{z}$, the direction of propagation. However, using $k = |k| e^{i\Omega}$

$$B = \frac{|k|}{\omega} e^{i\Omega} (\hat{z} \wedge E)$$

$E_{0x} = E_{0x,\text{real}} e^{i\theta}$

$e^{i(\alpha z - \omega t)} e^{i\theta}$

If the phase of E is $(\alpha z - \omega t + \theta)$

then the phase of $\underline{B}$ is $(\alpha z - \omega t + \Theta + \Omega)$

i.e. $\underline{E}$ and $\underline{B}$ are out of phase. At constant z $\underline{B}$ will be later than $\underline{E}$ reaching the same phase. $[-\omega t_E = -\omega t_B + \Omega \Rightarrow t_B = t_E + \Omega/\omega]$
Whereas at constant time $\underline{B}$ will reach the same phase before $\underline{E}$. $\left[\alpha z_E = \alpha z_B + \Omega \Rightarrow z_B = z_E - \frac{\Omega}{\alpha}\right]$

* A note of caution:

We have seen that v (k) are dependent on the frequency of the em wave $\Rightarrow$ dispersion. But we have blindly assumed that μ, ε, σ are all constants for a particular material.

This is not the case $\quad n = \frac{c}{v} = \left(\frac{\mu_0 \varepsilon_0 K_\mu K_e}{\mu_0 \varepsilon_0}\right)^{1/2}$

$$\text{e.g. } n_{water} = \sqrt{K_\mu K_e} = \sqrt{1 \times 80} \simeq 9$$

but you are familiar with the fact that n for water is about 1.33.

The problem lies in assuming $\mu\varepsilon$ ($K_\mu K_e$) are constant. The values $K_e = 80$, $K_\mu = 1$ are $\omega = 0$ values and do not apply at all frequencies.

This is the topic of 24-8 which is left for your own edification ...

[Continue with Chap 25 OR Chap 28 depending on what has already been covered]
$\Rightarrow$ p217 in notes

[Chapter 24 problems in conducting media 24-7, 11]