# Radiation – Chapter 28

* The material we have just completed, describing the behaviour of plane waves made up of oscillating $\underline{E}$ and $\underline{B}$ fields, did not address the issue of how such waves would be created. This is the subject of this chapter. We will show that oscillating charges "radiate" the EM waves whose properties we have already discussed.

Throughout the discussion of plane waves we have used solutions of Maxwell's equations in the form of the $\underline{E}$ and $\underline{B}$ fields themselves. As I indicated during our discussion of (Chap 22) potentials, following the overall discussion of Maxwell's equations, the phenomenon of radiation is more easily discussed in terms of these potentials, $\underline{A}$, $\phi$.

In fact, for the vacuum, we have, equivalent to Maxwell's eqn's

$$\underline{\nabla}^2 \underline{A} - \frac{1}{c^2}\partial^2 A/\partial t^2 = -\mu_0 \underline{J} \qquad (22\text{-}14)$$

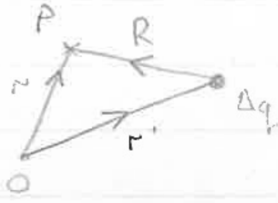
$$\nabla^2 \phi - \frac{1}{c^2}\partial^2\phi/\partial t^2 = -\rho/\varepsilon_0 \qquad -22\text{-}16)$$

plus the Lorentz condition $\underline{\nabla}\cdot\underline{A} + \frac{1}{c^2}\partial\phi/\partial t = 0$

where the fields $\underline{E}, \underline{B}$ are found from

$$\underline{B} = \underline{\nabla}\wedge\underline{A} \quad ; \quad \underline{E} = -\nabla\phi - \partial\underline{A}/\partial t$$

* Consider now a point charge $\Delta q = \rho\, d\tau'$ with the volume $d\tau'$ located at $\underline{r}'$




$\underline{R} = \underline{r} - \underline{r}'$

For the region <u>outside</u> $d\tau'$ $\rho = 0$ and we obtain

$$\nabla^2\phi - \frac{1}{c^2}\left(\partial^2\phi/\partial t^2\right) = 0$$

$\phi$ due to the point charge must have radial symmetry about $d\tau'$

$\Rightarrow \quad \phi = \phi(R,t)$

Now $\underline{\nabla}\phi = (\partial\phi/\partial R)\underline{R}/R$ [1-139]

and so $\nabla^2\phi = \underline{\nabla}\cdot\underline{\nabla}\phi = \underline{\nabla}\cdot\left[\frac{1}{R}\,\partial\phi/\partial R\ \underline{R}\right]$

$$= \underline{\nabla}\left(\frac{1}{R}\,\partial\phi/\partial R\right)\cdot\underline{R} + \frac{1}{R}\,\partial\phi/\partial R\ \underline{\nabla}\cdot\underline{R}$$

[1-139] and [1-141]
$$= \frac{1}{R}\,\partial^2\phi/\partial R^2\left(\underline{R}/R\right)\cdot\underline{R} - \partial\phi/\partial R\left(\underline{R}/R^3\right)\cdot\underline{R} + \frac{3}{R}\,\partial\phi/\partial R$$

$$= \partial^2\phi/\partial R^2 + \frac{2}{R}\,\partial\phi/\partial R$$

$$\Rightarrow \quad \frac{\partial^2\phi}{\partial R^2} + \frac{2}{R}\frac{\partial\phi}{\partial R} - \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) = 0$$

$$\text{or} \quad \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) - \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) = 0$$

[In other words we could have used $\nabla^2$ in spherical co-ordinates with R as the variable]

Solutions to this equation are spherical waves of the form

$$\phi(R,t) = \frac{\chi(R,t)}{R}$$

Where
$$\frac{\partial^2\chi}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0$$

and
$$\chi = \underbrace{f(R-ct)}_{\substack{\text{wave moving}\\ \text{towards } +R}} + \underbrace{g(R+ct)}_{\substack{\text{wave moving}\\ \text{towards } -R}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

Which in spherical terms means: Outgoing wave — Incoming wave.

or equivalently $f(t - R/c) + g(t + R/c)$

The form of $f$ and $g$ must be determined by the point source.
The behaviour of the source at time $t_0$ is mirrored by $f$ at $R$ at time $t = t_0 + R/c$ and by $g$ at $R$ at $t = t_0 - R/c$. That is, since "g" represents an incoming wave the wave must know what the source will do before it actually happens. This clearly is unsatisfactory — a violation of "cause and effect". Therefore we ignore the "g" solutions. [In quantum mechanics we cannot be so eager to ignore such solutions, since the Uncertainty Principle allows violation of causality for a limited time]

$$\Rightarrow \qquad \phi(R,t) = f\left(t - R/c\right) / R$$

$f$ is determined by the fact that at $R \simeq 0$ we must have

$$\phi = \frac{\Delta q(t)}{4\pi\varepsilon_0 R} \Rightarrow \phi = \frac{1}{4\pi\varepsilon_0 R}\rho\left(\underline{r}', t - R/c\right)\Delta\tau'$$

So that the net potential due to many charges is given by

$$\phi(\underline{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho\left(\underline{r}', t - R/c\right)}{R}\, d\tau'$$

this is the retarded potential, so called because it depends on the value of the charge at an earlier time — $t - R/c$. In other words it takes a finite time for an EM disturbance to travel through space.

* In a similar way it can be shown that

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\underline{J}\left(\underline{r}', t - R/c\right)}{R}\, d\tau'$$

N.B. It is worth noting the similarity of these solutions to the

static solutions
$$\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\underline{r}')\,d\tau'}{R}$$
$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\underline{J}(\underline{r}')\,d\tau'}{R}$$

* Similar to the static cases we now wish to expand these potentials in terms of a multipole expansion. We consider only the case of localised charges with harmonic variation in time.

In this case
$$\underline{J}(\underline{r}', t) = \underline{J}_0(\underline{r}')e^{-i\omega t} \quad ; \quad \rho(\underline{r}', t) = \rho_0(\underline{r}')e^{-i\omega t}$$
so that
$$\left.\begin{aligned}\phi(\underline{r}, t) &= \frac{e^{-i\omega t}}{4\pi\varepsilon_0}\int_{V'} \frac{e^{ikR}\rho_0(\underline{r}')\,d\tau'}{R} \\ \underline{A}(\underline{r}, t) &= \frac{\mu_0 e^{-i\omega t}}{4\pi}\int_{V'} \frac{e^{ikR}\,\underline{J}_0(\underline{r}')\,d\tau'}{R}\end{aligned}\right\} \begin{aligned} &t \to t - R/c \\ &\omega/c = k = 2\pi/\lambda\end{aligned}$$

The Lorentz condition $\underline{\nabla}\cdot\underline{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$

thus becomes $\underline{\nabla}\cdot\underline{A} = \frac{i\omega}{c^2}\phi$ (since only time dependence is $e^{-i\omega t}$)

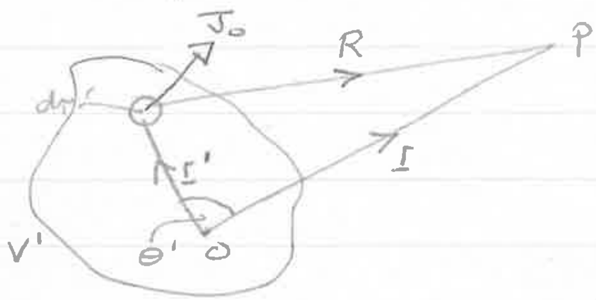
$$\Rightarrow \quad \phi = -\frac{ic^2}{\omega}(\underline{\nabla}\cdot\underline{A}) \qquad \text{——} \qquad (1)$$

and $\underline{E} = -\underline{\nabla}\phi - \partial A/\partial t$ becomes
$$\begin{aligned} &= +\nabla\left(\frac{ic^2}{\omega}\underline{\nabla}\cdot\underline{A}\right) + i\omega\underline{A} \\ &= \frac{ic^2}{\omega}\left[\underline{\nabla}\wedge(\nabla\wedge A) + \underline{\nabla}^2\underline{A}\right] + i\omega A \\ &= \frac{ic^2}{\omega}(\underline{\nabla}\wedge\underline{B}) + \frac{ic^2}{\omega}\underbrace{\left[\underline{\nabla}^2\underline{A} + \frac{\omega^2}{c^2}A\right]}_{\underline{\nabla}^2\underline{A} - 1/c^2\,\frac{\partial^2 A}{\partial t^2} = 0}\end{aligned}$$
where away from source $\underline{J} = 0$

$$\Rightarrow \quad \underline{E} = (ic/k)\, \underline{\nabla} \wedge \underline{B} \qquad — \qquad (2)$$

Thus for this situation we need only evaluate $\underline{A}$. $\phi$ will be found from (1), $\underline{B} = \underline{\nabla} \wedge \underline{A}$ and $\underline{E}$ from (2).

* Evaluation of $\underline{A}$.




Note similarity of diagram to that used for magnetic (electric) multipole expansion.

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int_{V'} \frac{e^{ikR}}{R} \underline{J}_0(\underline{r}')\, d\tau'$$

Remember
$$R = |\underline{r} - \underline{r}'| = \left(r^2 + r'^2 - 2\underline{r}\cdot\underline{r}'\right)^{1/2}$$
$$= r\left(1 - \frac{2\underline{r}\cdot\underline{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right)^{1/2} \simeq \underline{\underline{r\left(1 - \frac{\hat{\underline{r}}\cdot\underline{r}'}{r}\right)}}$$

but we are assuming that $\underline{r' \ll r}$ (small source volume and distant field point)

$$\Rightarrow \quad 1/R \simeq 1/r\left(1 - \frac{2\hat{\underline{r}}\cdot\underline{r}'}{r}\right)^{-1/2}$$
$$\simeq \underline{\underline{1/r\left(1 + (\hat{\underline{r}}\cdot\underline{r}')/r\right.}} \ldots \Big)$$

Higher order terms can be ignored if $k(\hat{\underline{r}}\cdot\underline{r}')$ is $\ll 1$ $\Rightarrow kr' \ll 1$ or $\lambda \gg r'$ ~~λ is much larger than the source size~~

Therefore
$$e^{ikR} = e^{ikr} e^{ik(R-r)} = e^{ikr} e^{-ik(\hat{\underline{r}}\cdot\underline{r}')}$$
$$= e^{ikr}\left[1 - (ik\hat{\underline{r}}\cdot\underline{r}') + \ldots\right]$$

and so
$$\frac{e^{ikR}}{R} = e^{ikr}\left(1 - ikr\hat{\underline{r}}\cdot\underline{r}'/r\right) 1/r \left(1 + (\hat{\underline{r}}\cdot\underline{r}')/r\right)$$

$$= \frac{e^{ikr}}{r}\left[1 + \left(\frac{\hat{\underline{r}}\cdot\underline{r}'}{r}\right)\boxed{1 - ikr} \ldots\right]$$

So that we may write

[Also $\frac{v}{c} = \frac{\omega r'}{c} = \frac{r'}{\lambda} \ll 1$ i.e. non relativistic
$v$ characteristic velocity of charge $= \omega r'$]

$$\underline{A}(\underline{r},t) = \underbrace{\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int_{V'}\underline{J}_0(\underline{r}')\,d\tau'}_{\underline{A}_{I}} + \underbrace{\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r^2}(1-ikr)\int_{V'}\underline{J}_0(\underline{r}')(\hat{\underline{r}}\cdot\underline{r}')\,d\tau'}_{\underline{A}_{II}}$$

Where we have extracted the terms dependent only on r from the integral over $d\tau'$.

These are the first <u>two</u> terms of the time dependent multipole expansion of $\underline{A}$

* We now investigate each of these terms in detail.

$A_I$ contains the integral $\int_{V'}\underline{J}_0(\underline{r}')\,d\tau'$. In order to evaluate it we make use of the vector relationship (1-129)

$$\oint_S \underline{B}(\underline{A}\cdot d\underline{a}) = \int_V \left[(\underline{A}\cdot\underline{\nabla})\underline{B} + \underline{B}(\underline{\nabla}\cdot\underline{A})\right] d\tau$$

putting $\underline{B} = \underline{r}'$ $\underline{A} = \underline{J}_0$ and $d\underline{a} = d\underline{a}'$ and using

$$(\underline{J}_0\cdot\underline{\nabla}')\underline{r}' = \underline{J}_0 \quad (1-121)$$

we have

$$\oint_{S'} \underline{r}'(\underline{J}_0\cdot d\underline{a}') = \int_{V'}\left[\underline{J}_0 + \underline{r}'(\underline{\nabla}'\cdot\underline{J}_0)\right] d\tau'$$

but $\underline{J}_0$ is restricted to a "small" volume and on $S'$ $\underline{J}_0 = 0$

$$\Rightarrow \int_{V'}\underline{J}_0\,d\tau' = -\int_{V'}\underline{r}'(\underline{\nabla}'\cdot\underline{J}_0)\,d\tau'$$

Note that in Ch 19 where we dealt with the static mag multipole expansion $\underline{\nabla}\cdot\underline{J} = 0$ $\Rightarrow \underline{\underline{\int_{V'}\underline{J}_0\,d\tau' = 0}}$ (No monopole term in mag multipole expansion)

Now the continuity relationship $\underline{\nabla}'\cdot\underline{J} = -\frac{\partial\rho}{\partial t}$

and since $\rho = \rho_0 e^{-i\omega t}$

$$\underline{\nabla}'\cdot\underline{J} = +i\omega\rho$$

and

$$\underline{\nabla}'\cdot\underline{J}_0 = i\omega\rho_0$$

$$\Rightarrow \int_{V'}\underline{J}_0\,d\tau' = -\int_{V'} i\omega\underline{r}'\rho_0\,d\tau'$$

$$\int_{V'} \underline{J}_0 \, d\tau' = -i\omega \underbrace{\int_{V'} \underline{r}' \rho_0(\underline{r}') \, d\tau'}_{\text{electric dipole moment of } \rho_0 = \underline{p}_0}$$

So that finally

$$\underline{A}_I = -\frac{i\mu_0\omega}{4\pi r}\left[\underline{p}_0 \, e^{i(kr-\omega t)}\right] \text{— oscillating dipole.}$$

which has the form of a field produced by a varying (in $t$ and $\underline{r}$) electric dipole $\Rightarrow$ hence the notation $\underline{A}_{ed}$

* The integral in $\underline{A}_{II}$ is $\displaystyle\int_{V'} \underline{J}_0(\underline{r}')(\hat{\underline{r}}\cdot\underline{r}')\,d\tau'$

This is identical to the integral we evaluated for the dipole term of the magnetic multipole expansion (p 219 Ch 19)

Adding and subtracting $\frac{1}{2}(\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'$ to the integrand above

$$\underline{J}_0(\underline{r}')(\hat{\underline{r}}\cdot\underline{r}') = \tfrac{1}{2}\left[(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0 - (\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'\right] + \tfrac{1}{2}\left[(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0 + (\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'\right]$$

(1-3a)

$$= \tfrac{1}{2}\left[(\underline{r}'\wedge\underline{J}_0)\wedge\hat{\underline{r}}\right] + \tfrac{1}{2}\left[(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0 + (\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'\right]$$

Thus $\underline{A}_{II}$ can be written as a sum of two parts

$$\underline{A}_{II} = \frac{\mu_0}{4\pi r^2}(1-ikr)e^{i(kr-\omega t)}\left[\frac{1}{2}\int_{V'}(\underline{r}'\times\underline{J}_0)\,d\tau'\right]\wedge\hat{\underline{r}}$$

$$+ \frac{\mu_0}{4\pi r^2}(1-ikr)e^{i(kr-\omega t)}\,\frac{1}{2}\int_{V'}\left[(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0 + (\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'\right]d\tau'$$

magnetic dipole moment of current distribution $\underline{J}_0$, $\underline{M}_0$ (19-19)

thus the first part of $\underline{A}_{II}$ is given by

$$\underline{A}_{md} = \frac{\mu_0}{4\pi r^2}(1-ikr)e^{i(kr-\omega t)}(\underline{M}_0\wedge\hat{\underline{r}})$$

i.e. the behaviour is the same as a varying magnetic dipole.

* The only remaining term to be evaluated is the second part of $\underline{A}_{II}$; in which the integrand is

$$(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0 + (\hat{\underline{r}}\cdot\underline{J}_0)\underline{r}'$$

In order to evaluate this expression we use the relations

$$\nabla'(\hat{\underline{r}}\cdot\underline{r}') = \hat{\underline{r}} \qquad (\hat{\underline{r}} \text{ is constant wrt } \nabla' \quad 1\text{-}113)$$

and $\underline{J}_0 = (\underline{J}_0\cdot\underline{\nabla}')\underline{r}'$ (1-121) so that the integrand becomes

$$(\hat{\underline{r}}\cdot\underline{r}')(\underline{J}_0\cdot\underline{\nabla}')\underline{r}' + \underline{r}'(\underline{J}_0\cdot\underline{\nabla}'(\hat{\underline{r}}\cdot\underline{r}'))$$

now $\quad \underline{\nabla}'\cdot(\underline{J}_0(\hat{\underline{r}}\cdot\underline{r}')) = \underline{J}_0\cdot\underline{\nabla}'(\hat{\underline{r}}\cdot\underline{r}') + (\hat{\underline{r}}\cdot\underline{r}')(\underline{\nabla}'\cdot\underline{J}_0)$

so that the integrand is

$$\boxed{(\underbrace{(\hat{\underline{r}}\cdot\underline{r}')\underline{J}_0}\cdot\underline{\nabla}')}\underline{r}' + \underline{r}'\left[\underline{\nabla}'\cdot\underbrace{(\underline{J}_0(\hat{\underline{r}}\cdot\underline{r}'))}\right] - \underline{r}'(\hat{\underline{r}}\cdot\underline{r}')(\underline{\nabla}'\cdot\underline{J}_0)$$

equal

And so using (1-129) again to convert a volume to a surface integral, the integral in question becomes

$$\oint_{s'} \underline{r}'(\underline{J}_0(\hat{\underline{r}}\cdot\underline{r}')\cdot d\underline{a}') - \int_{V'} i\omega\,\underline{r}'(\hat{\underline{r}}\cdot\underline{r}')\rho_0(\underline{r}')\,d\tau'$$

[using $\nabla'\cdot J = -\partial\rho/\partial t$ or $\nabla'\cdot J_0 = +i\omega\rho_0$]

Since $\underline{J}_0$ is non zero only within a "small" volume the surface integral is zero leaving as the second part of $\underline{A}_{II}$

$$-\frac{i\mu_0\omega(1-ikr)e^{i(kr-\omega t)}}{8\pi r^2}\int_{V'}\underline{r}'(\hat{\underline{r}}\cdot\underline{r}')\rho_0(\underline{r}')\,d\tau'$$

the $\hat{r}$ component of

Since $\underline{B} = \underline{\nabla}\wedge\underline{A}$ we can add a constant to $\underline{A}$ without changing $\underline{B}$ (or $\underline{E}$). Adding $-\int_{V'}\hat{\underline{r}}\ \tfrac{1}{3}(r')^2\rho_0(\underline{r}')\,d\tau'$ to the integrand we obtain

$$\underline{A}_{eq} = \frac{-i\omega\mu_0(1-ikr)e^{i(kr-\omega t)}}{24\pi r^2}\underbrace{\int_{V'}\left[3\underline{r}'(\hat{\underline{r}}\cdot\underline{r}') - (r')^2\hat{\underline{r}}\right]\rho_0(\underline{r}')\,d\tau'}_{Q\text{ — electric quadrupole vector}}$$

with components given by $\quad Q_\beta = \sum_{\alpha = x,y,z} l_\alpha Q_{\alpha\beta} \quad (\beta = x,y,z)$

where $\underline{\hat{r}} = l_x \underline{\hat{x}} + l_y \underline{\hat{y}} + l_z \underline{\hat{z}}$ .

Thus combining all these parts of $\underline{A}$ we have

$$\underline{A} = \underline{A}_{I} + \underline{A}_{II}$$
$$= \underset{\substack{\text{electric}\\ \text{dipole}}}{\underline{A}_{ed}} + \underset{\substack{\text{magnetic}\\ \text{dipole}}}{\underline{A}_{md}} + \underset{\substack{\text{electric}\\ \text{quadrupole}}}{\underline{A}_{eq}} \; \ldots$$

( E1    M1    E2    M2    E3 . . .

Remember these are only the first terms of the multipole expansion, we can expect each additional $\underline{A}$ (e.g. $\underline{A}_{III}$) to lead ~~too~~ two further contributions (e.g. M2 magnetic quadrupole, E3 - electric octopole).

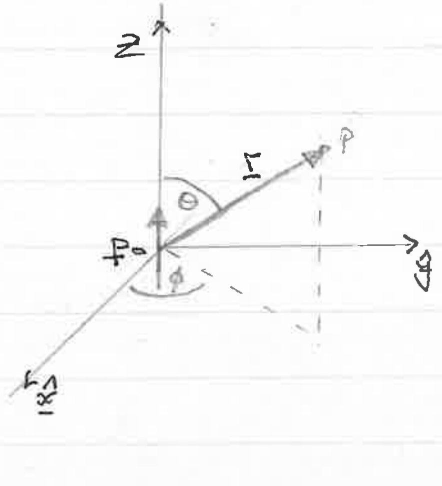
[Note that M1 and E2 have same r dependence $\propto 1/r^2$, similarly for M2, E3 etc]

Having obtained the multipole expansion of $\underline{A}$ we can now use the relations

$$\underline{B} = \underline{\nabla} \wedge \underline{A} \qquad \text{and} \qquad \underline{E} = i\frac{c}{k}(\underline{\nabla} \wedge \underline{B})$$

to evaluate the fields $\underline{B}$ and $\underline{E}$ for each term of the expansion

## Electric Dipole Radiation




Using spherical polar co-ordinates with the dipole $\underline{p}_o = p_o \underline{\hat{z}}$ we can write

$$\underline{A}_{ed} = -\frac{i\mu_o \omega p_o e^{i(kr-\omega t)}}{4\pi r} \underbrace{(\cos\theta\, \underline{\hat{r}} - \sin\theta\, \underline{\hat{\theta}})}_{\underline{\hat{z}}}$$

Using $\underline{B} = \underline{\nabla} \wedge \underline{A}$

magnitude of the charge.

* N.B (i) What I have evaluated here is the time-average power radiation. The text evaluates the instantaneous power – where the acceleration must be evaluated at the retarded time $(t = r/c)$.

(ii) In evaluating the radiated power we have assumed a harmonically oscillating charge. But the final result is independent of the frequency. In fact it can be shown that the result is true for a charge moving with any acceleration. [See if you can find out where]
(But remember this is NRel regime)

(iii) Emphasize: a charge only radiates if it is accelerated. Constant velocity charge does not radiate. A charge moving in a circular orbit is accelerated and therefore radiates, this is called synchrotron radiation. Minimising synchrotron radiation is an important aspect of accelerator construction. Classically, $a = v^2/r$ (uniform circular motion), thus when $v \sim c$ (as is true for electrons of a few MeV) $a$ decreases with increasing radius. The larger the radius of an accelerator the smaller the losses due to synchrotron radiation. Remember, any radiation losses must be provided by the mechanism of the accelerator. The CERN LEP accelerator 16 miles in circumference probably represents the final stage of circular electron accelerator technology. In other words to build a bigger accelerator would exceed the cost savings obtained by the reduced synchrotron radiation. Of course a full analysis must be done quantum mechanically for any real system. [Also note that some accelerators are built and used specifically to provide synchrotron radiation, since this radiation is a good source of X-rays]

Look up QM radiation

Assume a charge $q$ accelerating in the z-direction, then
$$a = \frac{d^2z}{dt^2} \quad \text{where } z \text{ is the position of the charge.}$$

Using $z = z_0 e^{-i\omega t}$ — harmonically oscillating charge
then $a = -\omega^2 z$.

But for a single charge $p = qz$ $\qquad \left[ p = \sum_{i=1}^{\infty} q_i r_i \right]$

$$\Rightarrow \quad p = -qa/\omega^2$$
$$p = -\left(\frac{q}{\omega^2}\right) a$$

For a harmonically oscillating charge, in the radiation zone we have
$$E = \frac{-k^2}{4\pi\varepsilon_0 r} \sin\theta \, p_0 e^{i(kr-\omega t)} \hat{\theta} \qquad c = \omega/k$$
$$= +\frac{\omega^2}{4\pi\varepsilon_0 c^2 r} \sin\theta \, \frac{q}{\omega^2} a_0 e^{i(kr-\omega t)} \hat{\theta} \quad \text{where } a_0 = -p_0 (\omega^2/q)$$
$$\underline{E} = \frac{q a_0 \sin\theta \, e^{i(kr-\omega t)}}{4\pi\varepsilon_0 c^2 r} \hat{\underline{\theta}}$$

Similarly we can show that $\underline{B} = \dfrac{q a_0 \sin\theta \, e^{i(kr-\omega t)}}{4\pi\varepsilon_0 c^3 r} \hat{\underline{\phi}}$

Therefore $\langle S \rangle = (1/2\mu_0) \, \text{Real} \left[ \underline{E} \wedge \underline{B}^* \right]$
$$= \frac{q^2 a_0^2 \sin^2\theta}{2\mu_0 . 16\pi^2 \varepsilon_0^2 c^5 r^2} \hat{\underline{r}} = \frac{q^2 a_0^2 \sin^2\theta}{32\pi^2 \varepsilon_0 r^2 c^3}$$

and the total power radiated is given by
$$\int \langle \underline{S} \rangle . d\underline{a} = \int_0^{2\pi}\int_0^{\pi} \frac{q^2 a_0^2}{32\pi^2 \varepsilon_0 c^3} \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$
$$= \underline{\underline{\frac{q^2 a_0^2}{12\pi\varepsilon_0 c^3}}}$$

Thus the power radiated by a moving charge is proportional to the square of the acceleration ($a_0$) and the square of the

$$\Rightarrow \quad R = \frac{2P}{I^2} = \frac{2\mu_0 \omega^4 \cancel{I^2} ds^2}{12\pi c \; \omega^2 \cancel{I^2}} = \frac{\mu_0 \omega^2}{6\pi c}(ds)^2$$

Using $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ ; $c = \omega/k$ ; $k = 2\pi/\lambda$ ; $c = 1/\sqrt{\mu_0 \varepsilon_0}$

$\omega^2 = c^2 k^2$

$\frac{\mu_0 \omega^2}{c} = \mu_0 c \frac{4\pi^2}{\lambda^2} = \frac{4\pi^2}{\lambda^2}\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} = \frac{4\pi^2}{\lambda^2} Z$

$$R = \underbrace{\tfrac{2}{3}\pi Z_0}_{790 \text{ ohms - for vacuum}} \left(\frac{ds}{\lambda}\right)^2$$

Maximising $R$, maximises the total power radiated (for a given $I$)

[N.B. Our original multipole expansion assumed that $\lambda \gg r'$ : $ds$ is of order $r'$ (size of source) $\Rightarrow \lambda \gg ds$. This limits the usefulness of maximising $R$]

* Remember, in all of the above we have been working in the "radiation zone" ($kr \gg 1$ ; $r \gg \lambda$). What happens to the Poynting vector when we do not make this approximation?

$\langle \underline{S} \rangle \propto \mathrm{Re}(\underline{E} \wedge \underline{B}^*)$

$$(\underline{E} \wedge \underline{B}^*) \propto \hat{\underline{r}} \left[\left(\frac{1}{kr} + \frac{i}{(kr)^2} - \frac{1}{(kr)^3}\right)\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)\right] \sin^2\theta - \hat{\underline{\theta}} \left[\frac{i}{(kr)^2} - \frac{1}{(kr)^3}\right]\left[\frac{1}{kr} - \frac{i}{(kr)^2}\right] 2\cos\theta\sin\theta$$

$$\propto \hat{\underline{r}} \sin^2\theta \left[\frac{1}{(kr)^2} + \frac{i}{(kr)^5}\right] - \hat{\underline{\theta}} \sin 2\theta \left[\frac{i}{(kr)^3} + \frac{i}{(kr)^5}\right]$$

Thus when we take the real part to obtain $\langle \underline{S} \rangle$ only one term remains, $\hat{\underline{r}} \frac{\sin^2\theta}{(kr)^2}$, which is exactly the term obtained in the radiation zone.

$\Rightarrow$ Time averaged energy flow $\langle \underline{S} \rangle$ is always obtained from the radiation zone contribution. The other contributions to $\underline{E} \wedge \underline{B}^*$ contribute to $\underline{S}$ (not time averaged) and therefore must represent energy which alternately flows towards and away from the source.

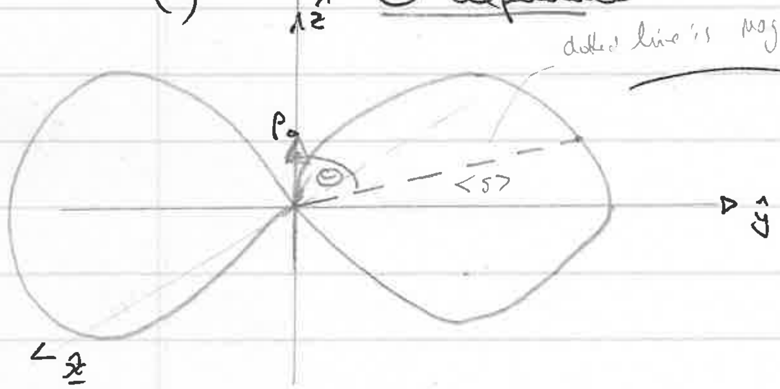
* <u>Accelerating Charges</u>

## Poynting Vector – Energy Flow (In radiation zone)

$$\langle \underline{S} \rangle = \frac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E} \wedge \underline{B}^*\right) = \frac{p_0^2 k^3 \mu_0 \omega \sin^2\theta}{2\mu_0 16\pi^2 \varepsilon_0 r^2}\,\hat{\underline{r}}$$

$$= \underbrace{\frac{p_0^2 \mu_0 \omega^4 \sin^2\theta}{32\pi^2 r^2 c}}_{\text{Magnitude of radiation intensity } \mathcal{I}}\,\hat{\underline{r}}$$

(i) $\theta$ dependence




$\longrightarrow$ $\theta$ dependence of $\mathcal{I}$

N.B. Maximum radiation occurs @ 90° to $p_0$ – there is no energy radiated along $p_0$. [Consequences in orientation of radio + TV antennas for transmission and reception]

(ii) $r$ dependence

Inverse square law $1/r^2$

(iii) $\omega$ dependence

$\langle S \rangle \propto \omega^4 \Rightarrow \langle S \rangle$ is larger for larger $\omega$. The higher the frequency the easier it is to radiate.

(iv) Total radiation rate

$$P = \int \langle \underline{S} \rangle \cdot d\underline{a} = \iint \mathcal{I}\, r^2 \sin\theta\, d\theta\, d\phi = \frac{p_0^2 \mu_0 \omega^4}{\pi^2 c\, 32}\, 2\pi \underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{4/3}$$

$$P = \frac{p_0^2 \mu_0 \omega^4}{12\pi c}$$

(v) "Radiation resistance"

Using $P = \frac{1}{2} I^2 R$ and assuming $I\,ds = dp/dt$

interpretation from $\underline{B}$ field – dipole is oscillating charge $\Rightarrow$ current $I$

$$I\,ds = -i\omega p_0 e^{-i\omega t}$$

$$\Rightarrow p_0^2 = I^2 ds^2/\omega^2$$

and $\underline{E} = \dfrac{p_0 e^{-i\omega t}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$

Using $\underline{p} = p_0 e^{-i\omega t}$ $\left( d\underline{p}/dt = -p_0 i\omega e^{-i\omega t} = -i\omega\underline{p}\right)$

$$\underline{B} = \frac{\mu_0 \sin\theta}{4\pi r^2}\left(\frac{dp}{dt}\right)\hat{\underline{\phi}} = \frac{\mu_0}{4\pi r^2}\left(d\underline{p}/dt\right)\wedge\hat{\underline{r}} \qquad \left[\hat{\underline{z}}\wedge\hat{\underline{r}} = \hat{\underline{\phi}}\sin\theta \text{ and } dp/dt\,\hat{\underline{z}} = d\underline{p}/dt\right]$$

and $\underline{E} = \dfrac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$

These fields are respectively the field produced by a current element $I\,d\underline{s} = d\underline{p}/dt$ $(\underline{B})$ (14-6 : Biot-Savart)

and that produced by a dipole $\underline{p}$ (at the origin) (see 8-50)

N.B. $\underline{p}$ oscillates with time but in the near zone the retardation effects are negligible $(e^{ikr} \simeq 1)$.

* Radiation Zone : $(kr \gg 1)$

At the opposite extreme $kr \gg 1 \Rightarrow r \gg \lambda$, the wavelength is very small compared to the distance at which we observe the fields, and

$$\underline{E} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r}\sin\theta\, e^{i(kr-\omega t)}\,\hat{\underline{\theta}}$$

$$\underline{B} = -\frac{\mu_0 \omega k p_0}{4\pi r}\sin\theta\, e^{i(kr-\omega t)}\,\hat{\underline{\phi}}$$

$\underline{B}, \underline{E}$ have the same $\theta$ and $r$ dependence, are at 90° to each other and vary according to $e^{i(kr-\omega t)}$ $\Rightarrow$ they comprise a plane wave, moving along $\hat{\underline{r}}$

$$\hat{\underline{r}}\wedge\underline{E} = -\frac{k^2 p_0 \sin\theta\, e^{i(kr-\omega t)}}{4\pi\varepsilon_0 r}\hat{\underline{\phi}} = \frac{\underline{B}k}{\varepsilon_0\mu_0\omega} = \underline{B}\,c$$

The radiation zone is more often than not the region of interest — e.g. transmission of radio waves, microwaves etc, therefore we consider some of the properties of this wave.

Maybe omit if short on time

$$\underline{B} = \frac{\hat{\underline{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\underline{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\underline{\phi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

but $A_\phi = 0$ and $\underline{A}$ has no $\phi$ dependence

$$\Rightarrow \underline{B} = +\frac{\hat{\underline{\phi}}}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r}e^{i(kr-\omega t)}\right) + \frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{r}e^{i(kr-\omega t)}\right]\right]\frac{i\mu_0\omega p_0}{4\pi}$$

$$= +\frac{i\mu_0\omega p_0}{4\pi r}\left[ik\sin\theta e^{i(kr-\omega t)} - \frac{\sin\theta}{r}e^{i(kr-\omega t)}\right]\hat{\underline{\phi}}$$

$$\underline{B} = -\frac{\mu_0\omega p_0 k^2}{4\pi}\left[\frac{1}{kr} + \frac{i}{k^2r^2}\right]\sin\theta e^{i(kr-\omega t)}\hat{\underline{\phi}}$$

Similarly we can show that (Remember $\underline{E} = i\frac{c}{k}\nabla\wedge\underline{B}$)

$$\underline{E} = -\frac{k^3 p_0}{4\pi\varepsilon_0}\left[\left[\frac{2i}{(kr)^2} - \frac{2}{(kr)^3}\right]\cos\theta\,\hat{\underline{r}} + \left[\frac{1}{kr} + \frac{i}{(kr)^2} - \frac{1}{(kr)^3}\right]\sin\theta\,\hat{\underline{\theta}}\right]e^{i(kr-\omega t)}$$

* Both $\underline{B}$ and $\underline{E}$ comprise outward travelling waves with speed $c = \omega/k$ and $\underline{E}$ $\underline{B}$ are at 90° to each other.

Remember these expressions are <u>exact</u>, except in so far that we have assumed that $r > r'$ (i.e. the field point is further than the dimensions of the source) and non-relativistic approx $\lambda \gg r'$ <u>and</u> the "real" values of $\underline{E}$, $\underline{B}$ are found by taking the <u>real</u> part of each expression.

We have already assumed $r \gg r'$ and $\lambda \gg r'$

We now simplify the two expressions by assuming that $r \ll \lambda \Rightarrow kr \ll 1$ (i.e. P is much closer to the source than $\lambda$ – the wavelength of the radiation)

* <u>Near zone</u> : $(kr \ll 1)$ $\left[e^{ikr} = 1 + ikr + \frac{(ikr)^2}{2!} + \ldots\right]$

$$\Rightarrow e^{ikr} \simeq 1$$

$$\underline{B} = -\frac{i\mu_0\omega\sin\theta\, p_0 e^{-i\omega t}}{4\pi r^2}\hat{\underline{\phi}}$$

* Magnetic Dipole Radiation

Starting from $\underline{A}_{md} = \dfrac{\mu_0 m_0}{4\pi r^2}(1 - ikr)e^{i(kr-\omega t)} \sin\theta\, \hat{\underline{\phi}}$

and proceeding in the same manner as the electric dipole we can show that

$$\underline{E}_{md} = -\frac{m_0}{p_0}\underline{B}_{ed} \qquad \underline{B}_{md} = \frac{m_0}{c^2 p_0}\underline{E}_{ed}$$

Or equivalently we can obtain the md fields by substituting (28-74, 28-75)

$p_0 \to \frac{m_0}{c}$ ; $E \to cB$ ; $B \to E/c$

These substitutions hold true for all regions (near, intermediate and radiation zone).

Also the time averaged Poynting Vector $\langle \underline{S} \rangle$ and the total radiated power flow are found from their electric dipole counterparts by substituting $p_0 = m_0/c$.

Magnetic dipole radiation is then seen to be proportional to $m_0^2$ (just as electric dipole is proportional to $p_0^2$). However, in the magnetic case there is an additional factor of $1/c^2$ — indicating that, all other things equal, magnetic dipole radiation is much weaker than electric dipole radiation. $\text{Power} = \frac{\mu_0 \omega^4 |m_0|^2}{12\pi c^3}$

* Electric Quadrupole Radiation:

I state, without proof, that for a linear quadrupole (i.e. a charge distribution with axial symmetry about the z-axis) we obtain electric and magnetic fields in the radiation zone ($kr \gg 1$)

$$\underline{E} = \frac{i\mu_0 \omega^3 Q_0}{32\pi c r} \sin 2\theta\, e^{i(kr-\omega t)}\, \hat{\underline{\theta}} \; ; \quad \underline{B} = |\underline{E}|/c \,.\, \hat{\underline{\phi}}$$

then $$\langle \underline{S} \rangle = \frac{\mu_0 \omega^6 |Q_0|^2}{2048\, \pi^2 c^3 r^2} \sin^2 2\theta\, \hat{\underline{r}}$$

$$\text{Total Power} = \frac{\mu_0 \omega^6 Q_0^2}{960\pi c^3}$$

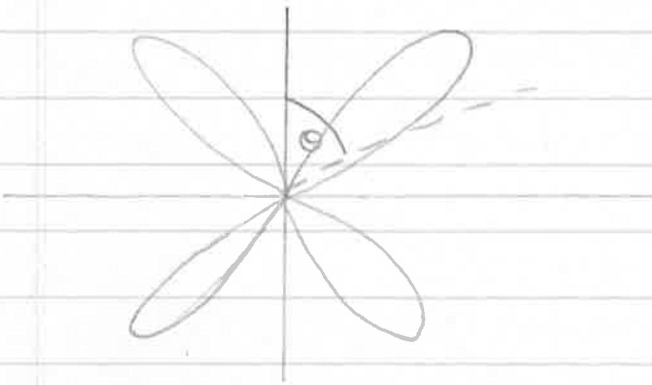
Important points

(i) $\langle \underline{S} \rangle$, Power $\propto \omega^6$

(ii) $\langle \underline{S} \rangle \propto 1/r^2$ (inverse square law)

(iii) $\underline{E}$, $\underline{B}$ form a transverse wave.

(iv) Angular dependence of $\langle \underline{S} \rangle$ is shown below ($\sin^2 2\theta$)




* Finally: You can guess the general behavior of some higher multipoles

$\propto 1/r^2$

e.g. M2 $\equiv$ E2 ($\langle S \rangle \propto \omega^6$, angular dependence $\sin^2 2\theta$)

E3 $\langle S \rangle$, Power $\propto \omega^8$

$\propto$

* Antennas — omit; at least for the time being

* Chap 28 : Problems — 1, 3, 5, 6, 7, 8, 9, 11, 13, 17