29. Electricity and Magnetism and Special Relativity

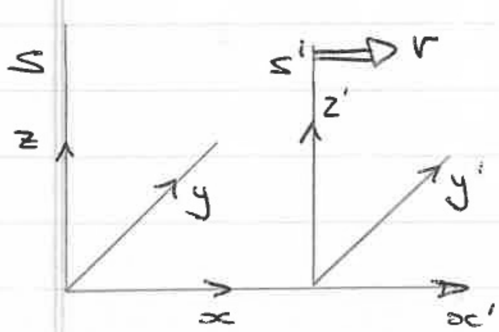
* I am going to assume that you have all covered the basics of special relativity. [e.g. Physics 300 or similar].
However, before applying the ideas of special relativity to E&M a brief review is in order.

* Einstein's Postulates

1) All inertial reference frames are equally suitable for the description of physical phenomena.
In other words, the laws of physics are the same in all inertial systems.
2) The speed of light (in vacuo) is the same for observers in all inertial reference frames, independent of the motion of the source.

[N.B. Inertial reference frames are reference frames moving with constant relative velocity]

* These assumptions lead to the Lorentz transformation (which replaces the intuitive Galilean transformation)



$$
\begin{array}{ll}
x' = \gamma(x - \beta c t) & = x - vt \\
y' = y & = y \\
z' = z & = z \\
t' = \gamma(t - \beta x/c) & = t
\end{array}
$$

Galilean transformation

and
$$x = \gamma(x' + \beta c t')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + \beta x'/c)$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\beta = v/c$
$(\gamma \geq 1)$ $(\beta \leq 1)$

[These transformations are for relative motion along x-axis - the co-ordinates transverse to the relative motion are unchanged]

* The Lorentz transformation leads to some unusual conclusions

(i) Simultaneity:

In S, 2 events at x_1 and x_2 are simultaneous, $t_1 = t_2$

Then in S'

$$\Delta t' = t_2' - t_1' = \gamma t_2 - \gamma\beta x_2/c - \gamma t_1 + \gamma\beta x_1/c$$
$$= (\gamma\beta/c)[x_1 - x_2] \neq 0$$

i.e. in S' these events are not simultaneous

(ii) Length Contraction:

In S' an object of length L' is at rest.

Then, at t = 0 (in S) we have

$$L' = \gamma L \quad \text{or} \quad L = L'/\gamma \Rightarrow L \leq L'$$

or the moving object $\left\{\begin{array}{c}\text{appears shorter}\\ \text{is contracted}\end{array}\right\}$ in S than its length in S'

(L' is called the proper length - the length in its rest frame)

(iii) Time dilation : clock at rest in S'

In S' the interval between two events, at $x' = 0$, is $\Delta t'$.

$$\Rightarrow \quad \Delta t = \gamma \Delta t'$$

$$\text{or} \quad \Delta t' = \Delta t / \gamma \quad \Rightarrow \quad \Delta t' \leqslant \Delta t$$

$\Rightarrow$ e.g. 1 hour in S' ($\Delta t'$) is less than the time Δt in S

In S' time has been stretched (dilated) which gives the fact that moving clocks run slow. For observer in S, clock is moving, and $\Delta t' \leqslant \Delta t \Rightarrow$ moving clocks run slow. [e.g. 2hrs pass in S - only 1hr in moving frame (in S')]

[$\Delta t'$ is the proper time - time measured in a clock's rest frame]

(iv) Velocity Addition :

$$u_x = dx/dt \quad ; \quad u'_x = dx'/dt'$$

$$\Rightarrow \quad u_x = \frac{\gamma(dx' + \beta c\, dt')}{\gamma(dt' + \beta\, dx'/c)} = \frac{u'_x + \beta c}{1 + \beta u'_x/c}$$

$$= \frac{u'_x + v}{(1 + u'_x v/c^2)}$$

[NOT simple $u = u' + v$]

Note that even if $u' = c$ and $v = c$ we get $u = c$ (relative velocities can never exceed c); and when $v \ll c$ we obtain the familiar Galilean transformation $u = u' + v$

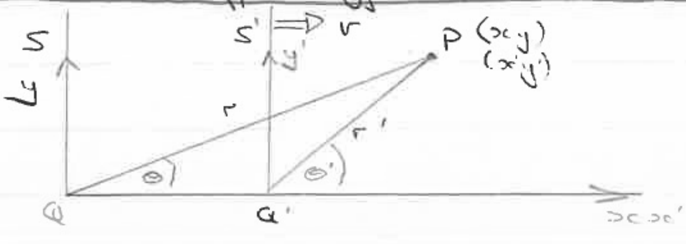
Also $\quad u_y = dy/dt \qquad u'_y = dy'/dt'$

$$= \frac{dy'}{\gamma(dt' + \beta\, dx'/c)} = \frac{u'_y}{\gamma(1 + v u'_x/c^2)}$$

and similarly for u_z, u'_z

i.e. the "transverse" velocity is also affected.

(v) Relativistic Doppler Effect and Aberration



Source is at O' in S'
P is observer at rest in S
" " " " S'

TIME DILATION

(1) $t' = \gamma(t - \beta x/c)$: $t = \gamma(t' + \beta x'/c)$ - (3)

(2) $x' = \gamma(x - \beta c t)$: $x = \gamma(x' + \beta c t')$ - (4)

$[\beta = v/c \quad , \quad \gamma = (1-\beta^2)^{-1/2}]$

Clock @ rest in S' @ $x' = 0$

$\Rightarrow$ Using (3) $t = \gamma t'$

$t \geqslant t'$

e.g. 1 hr in S' could be 2 hrs in S

For an observer in S, moving clocks run slow

$\Rightarrow$ Alternatively

Using (1) $t' = \gamma(t - \beta x/c)$

From (4) $x = \gamma(x' + \beta c t')$

or @ $x' = 0$ $x = \gamma \beta c t'$

$$\Rightarrow \quad t' = \gamma\left(t - \frac{\beta \gamma \beta \cancel{c} t'}{\cancel{c}}\right)$$

$$t' = \gamma t - \beta^2 \gamma^2 t'$$

$$t'(1 + \beta^2\gamma^2) = \gamma t$$

$$t'\left(1 + \frac{\beta^2}{1-\beta^2}\right) = \gamma t$$

$$t'\left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right) = \gamma t$$

$$t' \frac{1}{(1-\beta^2)} = \gamma t$$

$$t'\gamma^2 = \gamma t$$

$t = t'\gamma$ as above

Alternate Length Contraction Evaluation.

L'' @ rest in S' L' proper length

$$L^{*} = x_2 - x_1 = \gamma\left(x_2' - x_1' + \beta c\,(t_2' - t_1')\right)$$

but x_2 x_1 simultaneous in S $t_2 = t_1$.

$$t_2 - t_1 = \gamma\left(t_2' - t_1' + \frac{\beta}{c}(x_2' - x_1')\right)$$

$$0 = \quad "$$

$$t_2' - t_1' = -\frac{\beta}{c}(x_2' - x_1')$$

so that $x_2 - x_1 = \gamma\left(x_2' - x_1' - \beta^2(x_2' - x_1')\right)$

$$L = \gamma(L' - L'\beta^2)$$

$$L = L'\frac{\gamma}{\gamma^2} = \frac{L'}{\gamma}$$

spherical wave starting at origin

In S $\quad \psi = \dfrac{\psi_0}{r} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$ $\qquad$ In S' $\quad \psi' = \dfrac{\psi_0'}{r'} e^{i(\underline{k}'\cdot\underline{r}' - \omega' t')}$

and $\hat{k}$ is along $\underline{r}$ ($\underline{r}'$)

where $\omega = kc$ $\quad (\underline{k} = \hat{\underline{k}}k)$ $\qquad$ where $\omega' = k'c$

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$r = \frac{x^2}{r} + \frac{y^2}{r} = r$

and $r = x\cos\theta + y\sin\theta$ $\qquad$ and $r' = x'\cos\theta' + y'\sin\theta'$

Using Lorentz transformation $\qquad \underline{r}' = \hat{\underline{x}}\,\gamma(x - \beta ct)\cos\theta' + \hat{\underline{y}}\, y\sin\theta'$

Assuming that $(\underline{k}\cdot\underline{r} - \omega t) = (\underline{k}'\cdot\underline{r}' - \omega' t')$

we obtain

$$\omega\left(\frac{x\cos\theta + y\sin\theta}{c} - t\right) = \omega'\left(\frac{\gamma(x - \beta ct)\cos\theta' + y\sin\theta'}{c} - \gamma(t - \beta x/c)\right)$$

Since this must be true for all values of x, y at all t we must have coefficients of x, y, t equal independently.

Classical result is same expression without γ

$\Rightarrow$

1) $\quad \omega = \omega'\gamma(1 + \beta\cos\theta')$ $\qquad$ [t coefficient]

Source S' moving towards P in S $\Rightarrow \omega > \omega'$

Relativistic doppler effect formula

2) $\quad \omega\cos\theta = \omega'\gamma(\cos\theta' + \beta)$

which, dividing by 1) gives $\quad \cos\theta = (\cos\theta' + \beta)/(1 + \beta\cos\theta')$

Aberration formula

Note : When $\theta = \pi/2 \Rightarrow \cos\theta' = -\beta$

$$\Rightarrow \omega = \frac{\omega'(1 - \beta^2)}{(1 - \beta^2)^{1/2}} = \omega'\underbrace{(1 - \beta^2)^{1/2}}_{< 1}$$

$$\Rightarrow \omega < \omega'$$

i.e. there is a transverse doppler effect. (Non existent classically)

* All of the above phenomena have been experimentally verified. However, prior to the advent of special relativity a number of experiments were performed looking to identify a preferred reference frame which would be consistent with the Galilean

rather than Lorentz transformation.
e.g. Michelson - Morley and Trouton - Noble [p 497-499]

* But why did the Galilean transformation imply a preferred reference frame?
Start with the 1 Dim wave equation (which we know must be satisfied for a single component of $\underline{E}$ or $\underline{B}$)

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Now the Galilean transformation and Lorentz transformation are given by

$$x' = x - Vt \qquad\qquad x' = \gamma(x - \beta c t)$$
$$t' = t \qquad\qquad t' = \gamma(t - \beta x/c)$$

$$\Rightarrow \frac{\partial x'}{\partial x} = 1 \quad \frac{\partial t'}{\partial x} = 0 \; ; \qquad \frac{\partial x'}{\partial x} = \gamma \quad \frac{\partial t'}{\partial x} = -\frac{\gamma\beta}{c}$$

$$\frac{\partial x'}{\partial t} = -V \quad \frac{\partial t'}{\partial t} = 1 \; , \qquad \frac{\partial x'}{\partial t} = -\beta\gamma c \quad \frac{\partial t'}{\partial t} = \gamma$$

But $$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\psi}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\partial\psi/\partial x = \partial\psi/\partial x' \qquad\qquad \partial\psi/\partial x = \gamma\,\partial\psi/\partial x' - \gamma\beta/c\,\partial\psi/\partial t'$$

$$\partial^2\psi/\partial x^2 = \partial^2\psi/\partial x'^2 \qquad\qquad \partial^2\psi/\partial x^2 = \gamma^2\frac{\partial^2\psi}{\partial x'^2} + \frac{\gamma^2\beta^2}{c^2}\frac{\partial^2\psi}{\partial t'^2} - 2\frac{\gamma^2\beta}{c}\frac{\partial^2\psi}{\partial t'\partial x'}$$

and $$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial\psi}{\partial x'}\frac{\partial x'}{\partial t}$$

$$\partial\psi/\partial t = \partial\psi/\partial t' - V\,\partial\psi/\partial x' \qquad\qquad \partial\psi/\partial t = \gamma\,\partial\psi/\partial t' - \beta\gamma c\,\partial\psi/\partial x'$$

$$\partial^2\psi/\partial t^2 = \partial^2\psi/\partial t'^2 - 2V\,\partial^2\psi/(\partial t'\partial x') + V^2\,\partial^2\psi/\partial x'^2 \qquad \partial^2\psi/\partial t^2 = \gamma^2\frac{\partial^2\psi}{\partial t'^2} + \beta^2\gamma^2c^2\frac{\partial^2\psi}{\partial x'^2} - 2\beta\gamma^2c\frac{\partial^2\psi}{\partial t'\partial x'}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\partial^2\psi/\partial x'^2\,(c^2 - V^2) = \frac{\partial^2\psi}{\partial t'^2} - 2V\frac{\partial^2\psi}{\partial x'\partial t'} \qquad \partial^2\psi/\partial x'^2\left[\gamma^2c^2 - \beta^2\gamma^2c^2\right] - \partial^2\psi/\partial t'^2\left[\gamma^2 - \gamma^2\beta^2\right] = 0$$

Using wave eqn

$$\partial^2\psi/\partial x'^2 = \frac{1}{c^2}\,\partial^2\psi/\partial t'^2$$

Thus the Lorentz transformation leaves the form of the wave equation unchanged.
For the Galilean transformation if we try a solution of the form $\psi' = \psi_0' e^{ik'(x' - u't')}$ where u' is the wave velocity

then with $\partial^2\psi'/\partial x'^2 = -k'^2\psi'$; $\partial^2\psi'/\partial t'^2 = -k'^2u'^2\psi'$
and $\partial^2\psi'/\partial x'\partial t' = +k'^2u'\psi'$

then $-(c^2-V^2)\cancel{k'^2} = -\cancel{k'^2}u'^2 - 2V\cancel{k'^2}u'$

$$u'^2 + 2Vu' - (c^2-V^2) = 0 = [u+(c+v)][u-(c-v)]$$

$\Rightarrow u' = -(c+V)$ or $c-V$

these correspond to solutions for the wave travelling in $-x'$ and $+x'$ respectively.

* This shows that (i) a wave may have velocity $(u') > c$ and (ii) that there is only one frame in which Maxwell's equations have the form we originally assumed. (i.e. the frame in which $V=0$)

The special reference frame was assumed to be "fixed" relative to the "fixed stars." Note that the Lorentz transformation leaves the wave equation unchanged and is therefore consistent with Einstein's postulates of wave velocity always "c" and no prefered reference frame.

[Hand out "derivation" of Lorentz transformation]

* 4-Vectors and the generalised Lorentz Transformation:

By direct analogy with 3 dimensional vector analysis it is convenient to introduce the concept of a 4-vector, where the 4^{th} component is given by ict.

Then a 4-vector in space can be represented by

$$(x, y, z, ict) \quad \text{or} \quad (\underline{r}, ict)$$

[Note that the 4^{th} component $= ict$ is different from notation those used in particle physics; where we use the covariant/contravariant notation and did not need imaginary components]

[But since most/all of you have not taken particle physics, I will stick with the text notation – <u>Remember</u> it is only a matter of notation – which notation you use will not alter the results you get]

The "time" component must be imaginary since we demand that

$$x^2 + y^2 + z^2 - c^2t^2 = \underbrace{s^2}_{\substack{\text{invariant} \\ \text{under LT's}}} = x'^2 + y'^2 + z'^2 - c^2t'^2$$

That s must be invariant can easily be seen by considering an EM wave emitted from the origin when the origins of S and S' coincide. Then since EM waves travel at c – regardless of the frame – we must have $x^2 + y^2 + z^2 = c^2t^2$ in S and $x'^2 + y'^2 + z'^2 = c^2t'^2$ in S' $\Rightarrow$ s is invariant.

* With the notation $x_1 = x$; $x_2 = y$; $x_3 = z$; $x_4 = ict$ we may write

$$s^2 = \sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} x_i'^2$$

and the Lorentz transformation becomes

$$x_j' = \sum_{i=1}^{4} a_{ji} x_i \quad (j = 1,2,3,4)$$

$$x_j' = a_{ji} x_i \quad \text{(where repeated indices} \Rightarrow \text{summation)}$$

where a_{ji} are the 16 components of a 4×4 matrix.

In the most general case, where we are not able to define the relative motion of S and S' as an axis, non of the 16 elements are zero. However, if we define the x-axis as the direction of S and S' relative motion we obtain

$$a_{ji} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

So that
$$x_1' = \gamma x_1 + i\beta\gamma x_4 \qquad [x' = \gamma(x - \beta ct)$$
$$x_2' = x_2 \qquad y' = y$$
$$x_3' = x_3 \qquad z' = z$$
$$x_4' = \gamma x_4 - i\beta\gamma x_1 \qquad \underbrace{t' = \gamma(t - \beta x/c)}_{\text{simple LT we have already seen}}]$$

* It can easily be shown that the scalar (inner) product of two 4-vectors is invariant

$$\left[\sum_{i=1}^{4} A_i' B_i' = \sum_i \left(\sum_\mu a_{i\mu} A_\mu\right)\left(\sum_\lambda a_{i\lambda} B_\lambda\right) \right.$$
$$= \sum_\mu \sum_\lambda A_\mu B_\lambda \left(\sum_i a_{i\mu} a_{i\lambda}\right)$$

But since $\sum_i x_i'^2 = \sum_i \left(\sum_\mu a_{i\mu} x_\mu\right)\left(\sum_\lambda a_{i\lambda} x_\lambda\right) = \sum_\mu \sum_\lambda \underbrace{\left(\sum_i a_{i\mu} a_{i\lambda}\right)}_{\delta_{\mu\lambda}} x_\mu x_\lambda$

$= \sum_i x_i^2$ then

$$\left. \Rightarrow \sum_{i=1}^{4} A_i' B_i' = \sum_\mu \sum_\lambda A_\mu B_\lambda \delta_{\mu\lambda} = \sum_{i=1}^{4} A_i B_i \right]$$

* We are of course not limited to the spacial and time co-ordinates defining a 4-vector. Any quantity that transforms in the same way as x_i is a 4-vector.

e.g. $(\underline{p}, iE/c)$ $(p_x\ p_y\ p_z\ iE/c)$ is known as the 4-momentum, from which we obtain the relationship

$$p^2 - E^2/c^2 = -M^2c^2$$
$$(p^2c^2 + M^2c^4 = E^2)$$

Where M is the "effective mass" (or rest mass for a single particle) and is invariant under the Lorentz

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & & & \\ a_{31} & & & \\ a_{41} & & & \end{bmatrix} \begin{bmatrix} a_{11} & & & \\ a_{21} & & & \\ a_{31} & & & \\ a_{41} & & & \end{bmatrix}$$

Standard Matrix Multiply

Top left element $= a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} + a_{14}a_{41}$

(29-64) from LTx2

But $\sum_\lambda a_{\lambda\mu} a_{\lambda\nu} = a_{1\mu}a_{1\nu} + a_{2\mu}a_{2\nu} + a_{3\mu}a_{3\nu} + a_{4\mu}a_{4\nu}$

$\delta_{11} = a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31} + a_{41}a_{41}$

$= \gamma^2 - \gamma^2\beta^2 = \underline{1}$

Not standard Matrix Mult.

$\sum_\lambda a_{\lambda\mu} a_{\nu\lambda} = a_{1\mu}a_{\nu 1} + a_{2\mu}a_{\nu 2} + a_{3\mu}a_{\nu 3}$

$(1,1) = a_{11}a_{11} + a_{21}a_{12} + a_{31}a_{13} + \ldots$

transformation.

* A detailed study of relativistic particle mechanics is not appropriate at this stage. Suffice to say that the concept of 4-momentum is very powerful

4-momentum conservation combines both momentum and energy conservation

Some important results are indicated below

1) $\underline{\text{4 - velocity}} \;=\; \left[\gamma v_x ,\ \gamma v_y ,\ \gamma v_z ,\ i\gamma c\right] = \gamma(\underline{v}, ic)$

2) $\underline{\text{4 - Momentum}} \;=\; m_0 \gamma(\underline{v}, ic) \;=\; (\underline{p},\ iE/c)$

$$\underbrace{\underline{p} = m_0 \gamma \underline{v}} \qquad E = \gamma m_0 c^2$$

By direct analogy with 3 momenta ($\underline{p} = m\underline{v}$) we may call $m_0\gamma$ ($=m$) the "relativistic mass" and since γ increases as v increases we see the "increase" of mass with increasing v.

3) The $E = \gamma m_0 c^2$ by expansion of $\gamma = 1/(1 - v^2/c^2)^{1/2}$

$$= m_0 c^2 \left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \ldots\right)$$

$$= m_0 c^2 + \tfrac{1}{2} m_0 v^2 + \ldots$$

$$E = m_0 c^2 + T$$

Total energy = Rest mass energy + kinetic energy

NB: Relativistic Kinetic Energy (T) is not $\tfrac{1}{2}mv^2$, but a sum of terms which in the limit $v \ll c$ reduces to the non-relativistic limit of $\tfrac{1}{2}m_0 v^2$.

4) Note that the concept of force plays a less central role.

A 4-force F_μ can be defined by [Minkowski force]

$$F_\mu = \frac{dP_\mu}{d\tau}$$

where P_μ is the 4-momentum $(\underline{p}, iE/c)$

and $d\tau$ is the element of proper time measured in the object's rest frame. [$d\tau$ is therefore invariant].

Now $\gamma\, d\tau = dt$

$$\Rightarrow \quad F_\mu = \gamma\, dp_\mu / dt$$

$$F_\mu / \gamma = \underbrace{d/dt\,(m_o \gamma \underline{v}}_{\underline{F}}, \, i m_o \gamma c)$$

$$\Rightarrow \quad F_\mu = \gamma \underline{F} \qquad \text{(for 1}^{st}\text{ 3 components of } F_\mu)$$

5) The 4-momenta undergoes the Lorentz transformation exactly as any other 4-vector.

e.g.
$$p_x = \gamma(p_x' + \beta E'/c); \quad p_x' = \gamma(p_x - \beta E/c)$$
$$p_y = p_y'$$
$$p_z = p_z'$$
$$E = \gamma(E' + \beta c p_x') \quad ; \quad E' = \gamma(E - \beta c p_x)$$

[for a LT along the x-axis]

[Here assign 1st Ch 29 hw]

* So what about electromagnetism and its formalism in relativistic terms?

All our non-relativistic discussion has relied heavily on the mathematical constructs of div, grad and curl – thus before describing EM in relativistic terms we need 4-vector equivalences of these constructs.

* Gradient:

$$\underline{\nabla} = \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$$

perhaps the 4-vector notation will look like

$$\Box = \hat{x}_1 \frac{\partial}{\partial x_1} + \hat{x}_2 \frac{\partial}{\partial x_2} + \hat{x}_3 \frac{\partial}{\partial x_3} + \hat{x}_4 \frac{\partial}{\partial x_4}$$

the components of $\Box$ are $\left[\partial/\partial x, \partial/\partial y, \partial/\partial z, -\frac{i}{c}\partial/\partial t\right]$

Now from calculus (chain rule)

$$\frac{\partial}{\partial x'_\mu} = \sum_\nu \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu}$$ for x_ν a function of x'_μ

but if x'_μ and x_μ are related by the Lorentz transformation

$$x_\nu = \sum_\mu a_{\nu\mu} x'_\mu \Rightarrow \frac{\partial x_\nu}{\partial x'_\mu} = a_{\nu\mu}$$

$$\Rightarrow \left(\frac{\partial}{\partial x'_\mu}\right) = \sum_\nu a_{\nu\mu}\left(\frac{\partial}{\partial x_\nu}\right)$$

but this is exactly the transformation property of a 4-vector.

$\Rightarrow$ $\Box$ is a 4-vector operator (4-gradient).

↳ d'Alembertian

* Divergence: $\underline{\nabla}\cdot\underline{A} = \frac{\partial A_{x_1}}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \sum_{\mu=1}^{3}\frac{\partial A_\mu}{\partial x_\mu}$

perhaps $\underline{\Box}\cdot\underline{A} = \sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial x_\mu}$

is this invariant? Is $\sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \frac{\partial A_i}{\partial x_i}$?

Where we know

that $A'_i = \sum_\nu a_{i\nu} A_\nu$ and $\partial/\partial x'_i = \sum_\mu a_{i\mu}\,\partial/\partial x_\mu$

$$\Rightarrow \sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_\mu\sum_\nu\sum_i a_{i\mu}\frac{\partial}{\partial x_\mu}(a_{i\nu}A_\nu)$$

$$= \sum_i\sum_\mu\sum_\nu a_{i\mu}a_{i\nu}\frac{\partial A_\nu}{\partial x_\mu}$$

But $\underbrace{\sum_i a_{i\mu}a_{i\nu} = \delta_{\mu\nu}}_{\text{from notes p 248}}$

and so $\sum_{\mu} \frac{\partial A'_\mu}{\partial x'_\mu} = \sum_{\nu} \frac{\partial A_\nu}{\partial x_\nu}$

$\underbrace{\qquad\qquad\qquad}$ $\underline{\square} \cdot \underline{A}$ is invariant under L.T.

* Curl : $\underline{\nabla} \wedge \underline{A}$ — this is more awkward to handle. However, it can be shown that the curl is an anti-symmetric tensor. (2nd rank – since we require 2 indices. Vector is a 1st rank tensor, scalar is a zero rank tensor)

$$C_{\mu\nu} = \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right)$$

N.B. There are 16 elements to this tensor, but since it is antisymmetric – $C_{\mu\nu} = -C_{\nu\mu}$ – all diagonal elements must be zero, leaving only 6 independent components. On the other hand a symmetric tensor has non-zero diagonal elements leading to 10 independent components.

For a second rank tensor under Lorentz Transformation. $C'_{\mu\nu} = \sum_\lambda \sum_\rho a_{\mu\lambda} a_{\nu\rho} C_{\lambda\rho}$

* $\underline{\nabla}^2$: $\underline{\nabla}^2 = \sum_{\mu=1}^{3} \frac{\partial^2}{\partial x_\mu^2}$

In terms of 4-vectors $\underline{\nabla}^2 \Rightarrow \square^2 = \sum_{\mu=1}^{4} \frac{\partial^2}{\partial x_\mu^2} = \underbrace{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\text{d'Alembertian}}$

$\square^2$ is invariant under LT.

* <u>Electromagnetism in 4-vector notation</u>

Equation of continuity $\underline{\nabla} \cdot \underline{J} = -\partial \rho / \partial t$

becomes $\underline{\sum_{\mu=1}^{4} \frac{\partial J_\mu}{\partial x_\mu} = 0}$ $(\Box \cdot J = 0)$ — (R1)

Where $J_\mu = (J_x, J_y, J_z, ic\rho) = (J_1\, J_2\, J_3\, J_4)$

[note that 4th term in summation above is $\frac{\partial J_4}{\partial x_4} = -\frac{i}{c}\frac{\partial}{\partial t}(ic\rho) = \frac{\partial \rho}{\partial t}$]

Equation R1 is automatically relativistically invariant so long as J_μ is a 4-vector.

But is J_μ a 4-vector?

Assume that charge is the same in any reference frame then $\rho\, d\tau = \underbrace{\rho_0\, d\tau_0}_{\text{charge density and volume in rest frame.}}$ (τ is volume here not time)

but $\gamma\, d\tau = d\tau_0$ (length along one axis is contracted)

$\Rightarrow \rho = \gamma \rho_0$ (charge density ρ is increased in moving frame)

But $\underline{J} = \rho \underline{v}$ (From Chap 12)

$$\left.\begin{aligned} \underline{J} &= \gamma \rho_0 \underline{v} \\ \text{and}\quad ic\rho &= i\gamma\rho_0 c \end{aligned}\right\} J_\mu = \rho_0 \underbrace{\gamma(\underline{v}, ic)}_{\text{4 velocity}}$$

$\Rightarrow$ J_μ is a 4-vector — the <u>4-current</u> and R1 is the relativistic formulation of the equation of continuity.

* Remember, in terms of potentials we could write Maxwell's eqns

$$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{J} \qquad (\Box^2 A = -\mu_0 J)$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0 \qquad (\Box^2 \phi = -\rho/\varepsilon_0)$$

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \quad \text{— Lorentz condition.}$$

With $A_\mu = \left(A_x \; A_y \; A_z \; \frac{i\phi}{c} \right) = \left(A_1 \, A_2 \, A_3 \, A_4 \right)$
and using our definition of $\underline{\Box} \cdot \underline{A}$ we find that the
Lorentz Condition reduces to

$$\underline{\Box} \cdot \underline{A} = 0 \qquad \text{or} \qquad \sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = 0 \qquad \text{(R2)}$$

Similarly it can be shown that

(R3) $$\underline{\Box}^2 A_\mu = -\mu_0 J_\mu$$ can represent the first two equations

[Since $\Box^2$ is invariant under LT and J_μ is a 4-vector
A_μ is also a 4-vector — the 4-potential]

(R1) (R2) and (R3) are in principle all we need for relativistic electromagnetism.

* What about the $\underline{E}$ and $\underline{B}$ fields?
These are found by using

$$\underline{B} = \underline{\nabla} \wedge \underline{A} \quad \text{and} \quad \underline{E} = -\underline{\nabla}\phi - \partial \underline{A}/\partial t.$$

in 3 dimensions.
Thus we expect that

Electromagnetic field tensor $$f_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$ [4 dimensional curl of A]

will be helpful.
Now since the curl is anti-symmetric all diagonal elements of $f_{\mu\nu}$ are zero.

$$f_{12} = \frac{\partial A_{2_y}}{\partial x_{1_x}} - \frac{\partial A_{1_x}}{\partial x_{2_y}} = B_z \quad (B_3)$$

Similarly for all other f's with indices $\mu,\nu = 1,2,3$

Now $$f_{41} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} \qquad x_4 = ict \qquad A_4 = i\phi/c$$

$$= \frac{1}{ic}\frac{\partial A_x}{\partial t} - \frac{i}{c}\frac{\partial \phi}{\partial x} = \frac{i}{c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right]$$

$$= \frac{iE_x}{c} = \frac{iE_1}{c}$$

and similarly for all other $f_{\mu\nu}$ components involving $\mu, \nu = 4$

$$\Rightarrow \quad f_{\mu\nu} = \begin{array}{cc} & \begin{array}{cccc} 11 & 12 & 13 & 14 \end{array} \\ \begin{array}{c} 11 \\ 21 \\ 31 \\ 41 \end{array} & \begin{bmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{bmatrix} \\ & \begin{array}{cccc} & 42 & 43 & \end{array} \end{array}$$

* A_μ and J_μ are 4-vectors, their transformation properties are exactly equivalent to all other 4-vectors under LT

e.g.
$$A_1' = \gamma(A_1 + i\beta A_4)$$
$$A_x' = \gamma(A_x - V\phi/c^2)$$

But What about $\underline{E}$ and $\underline{B}$ — they are components of the electromagnetic field tensor $f_{\mu\nu}$ and it is the tensor whose components transform as

$$f_{\mu\nu}' = \sum_{\lambda\rho} a_{\mu\lambda} a_{\nu\rho} f_{\lambda\rho}$$

e.g. $$f_{43}' = \frac{iE_z'}{c} = \sum_{\lambda\rho} a_{4\lambda} a_{3\rho} f_{\lambda\rho}$$

(29-77)

$$a = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \sum_{\lambda} a_{4\lambda} a_{33} f_{\lambda 3} \qquad [a_{31} = a_{32} = a_{34} = 0]$$

(29-77)

$$= a_{41} a_{33} f_{13} + a_{44} a_{33} f_{43}$$

$$\frac{iE_z'}{c} = +i\beta\gamma B_y + \frac{i\gamma E_z}{c}$$

$$\Rightarrow E_z' = \gamma(E_z + \beta c B_y)$$

Similarly for all the other components of $\underline{E}$, $\underline{B}$

[E_x' and E_y' are evaluated in text at bottom of page 521]

$E_x' = E_x$; $E_y' = \gamma(E_y - \beta c B_z)$

* In general, we may write, with $\underline{E} = \underline{E}_{\parallel} + \underline{E}_{\perp}$; $\underline{B} = \underline{B}_{\parallel} + \underline{B}_{\perp}$

$$\underline{E}_{\parallel}' = \underline{E}_{\parallel} \qquad \underline{B}_{\parallel}' = \underline{B}_{\parallel}$$

$$\underline{E}_{\perp}' = \gamma(\underline{E}_{\perp} + \underline{v}\wedge\underline{B}_{\perp}) \qquad \underline{B}_{\perp}' = \gamma\left(\underline{B}_{\perp} - \frac{(\underline{v}\wedge\underline{E}_{\perp})}{c^2}\right)$$

and inverses $\underline{E}_{\perp} = \gamma(\underline{E}_{\perp}' - \underline{v}\wedge\underline{B}_{\perp}')$ $\qquad \underline{B}_{\perp} = \gamma\left(\underline{B}_{\perp}' + \frac{\underline{v}\wedge\underline{E}_{\perp}'}{c^2}\right)$

where $\underline{E}_{\parallel}$ and $\underline{B}_{\parallel}$ are components of $\underline{E}$ and $\underline{B}$ along the direction of $\underline{v}$ (relative velocity of S and S') and $\underline{E}_{\perp}$ and $\underline{B}_{\perp}$ are @ 90° to $\underline{v}$

N.B.

1) The $\parallel$ components are unchanged cf with length contraction.

2) When $v \ll c$, non-relativistic limit, $(\gamma \sim 1)$

$$\underline{B}' = \underline{B}$$

$$\underline{E}' \sim \underline{E} + \underline{v}\wedge\underline{B}_{\perp}$$

$$\underline{E} + \underline{v}\wedge(\underline{B} - \underline{B}_{\parallel})$$

$$\underline{E} + \underline{v}\wedge\underline{B} \qquad (\underline{v}\wedge\underline{B}_{\parallel} = 0)$$

$\underline{v}$, $\underline{B}_{\parallel}$ are @ 0° by definition

[Chap 17 Faraday's Law (17-29)]

3) If $\underline{B} = 0$ $\underline{E} \neq 0$ in S

then $\underline{E}' = \underline{E}_{\perp}' + \underline{E}_{\parallel}' = \underline{E}_{\parallel} + \gamma\underline{E}_{\perp}$

and $\underline{B}' = \underline{B}_{\perp}' + \underline{B}_{\parallel}' = -\frac{\gamma}{c^2}(\underline{v}\wedge(\underline{E} - \underline{E}_{\parallel})) = -\frac{\gamma}{c^2}(\underline{v}\wedge\underline{E})$

$$= -\frac{\gamma}{c^2}\frac{(\underline{v}\wedge\underline{E}_{\perp}')}{\gamma} = -\frac{\underline{v}\wedge\underline{E}'}{c^2}$$

i.e. in S' a $\underline{B}'$ field appears whose magnitude is given by $\frac{E'v}{c^2}$.

4) If $\underline{E} = 0$ $\underline{B} \neq 0$ in S

then $\underline{E}' = \underline{E}'_\perp + \underline{E}'_{\parallel} = \gamma(\underline{v} \wedge \underline{B}_\perp) = (\underline{v} \wedge \underline{B}'_\perp) = (\underline{v} \wedge \underline{B}')$

$\underline{B}' = \underline{B}'_\perp + \underline{B}'_{\parallel} = \gamma \underline{B}_\perp + \underline{B}_{\parallel}$

i.e. In S' an $\underline{E}'$ field appears with magnitude vB'

*

So far we have written the equation of continuity, Maxwells equations in terms of $\underline{A}$ and ϕ and obtained the electromagnetic field tensor $f_{\mu\nu}$ all of which are covariant under Lorentz transformations.

Thus we may conclude that Maxwell's equations are indeed Lorentz invariant and need no adjustment when dealing with relativistic speeds.

Nevertheless it would be nice to obtain the Maxwell equations themselves in covariant form.

[Invariant — magnitude of a quantity is unchanged under Lorentz transformation

Covariant — "Form" of an equation is unaltered by LT
e.g. $F + G \to F' + G'$]

*

$$\left.\begin{aligned} \underline{\nabla} \wedge \underline{B} &= \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \\ \underline{\nabla} \cdot \underline{E} &= \rho/\varepsilon_0 \end{aligned}\right\} \quad \sum_\nu \frac{\partial f_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \qquad \text{— } (R_4)$$

$$\left.\begin{array}{l}\underline{\nabla}\cdot\underline{B} = 0 \\ \underline{\nabla}\wedge\underline{E} = -\partial\underline{B}/\partial t\end{array}\right\} \quad \frac{\partial f_{\lambda\rho}}{\partial x_\nu} + \frac{\partial f_{\rho\nu}}{\partial x_\lambda} + \frac{\partial f_{\nu\lambda}}{\partial x_\rho} = 0 \qquad \text{(R5)}$$

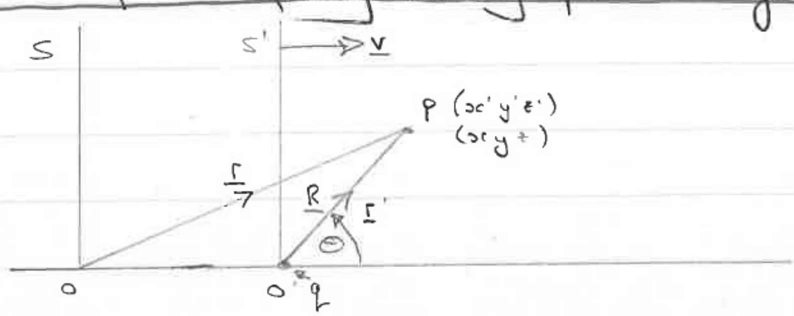
Using $c = 1/\sqrt{\mu_0\varepsilon_0}$ it is relatively easy to show that (R4) leads to the associated pair of equations. [$\mu = 4$ gives $\underline{\nabla}\cdot\underline{E} = \rho/\varepsilon_0$, $\mu = 1,2,3$ are the 3 components of $\underline{\nabla}\wedge\underline{B}$].

That (R5) leads to $\underline{\nabla}\cdot\underline{B} = 0$ and $\underline{\nabla}\wedge\underline{E} = -\partial\underline{B}/\partial t$ is somewhat more tedious. However it should be noted that although $\lambda\rho\nu$ can each take on all 4 values 1,2,3,4 terms where $\lambda = \rho = \nu$ are all zero (since $f_{\nu\nu} = 0$). Also since $f_{\mu\nu} = -f_{\nu\mu}$ all terms where one pair of indices are equal are zero. In the end we are left with only 4 terms

$$\begin{array}{ccccc} \lambda & 1 & 2 & 3 & 4 \\ \rho & 2 & 3 & 4 & 1 \\ \nu & 3 & \underbrace{4}_{} & \underbrace{1}_{} & \underbrace{2}_{} \end{array}$$

$$\underline{\nabla}\cdot\underline{B} = 0 \qquad \underline{\nabla}\wedge\underline{E} = -\partial\underline{B}/\partial t.$$

* Field of a uniformly moving point charge



q at O' in S'

In S' $\quad \underline{E}' = \dfrac{q\underline{r}'}{4\pi\varepsilon_0 r'^3}$ $\qquad \underline{B}' = 0$

now $r'^2 = x'^2 + y'^2 + z'^2$

$= \left(\gamma(x - vt)\right)^2 + y^2 + z^2$

In $\underline{S}$ position vector of P w.r.t O' is $\underline{R}$ where

$$\underline{R} = \hat{\underline{x}}(x-vt) + y\hat{\underline{y}} + z\hat{\underline{z}}$$

Using the Lorentz transformation properties of $\underline{E}'$ and $\underline{B}'$

$$E_x = E_x' = \frac{qx'}{4\pi\varepsilon_0 r'^3} = \frac{q\gamma(x-vt)}{4\pi\varepsilon_0\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

$$= \frac{q\gamma R_x}{4\pi\varepsilon_0\left[\gamma^2 R_x^2 + R_y^2 + R_z^2\right]^{3/2}}$$

Using inverse transformations (p256)

Similarly $E_y = \gamma E_y' = \dfrac{\gamma q y'}{4\pi\varepsilon_0 r'^3} = \dfrac{\gamma q R_y}{4\pi\varepsilon_0\left[\gamma^2 R_x^2 + R_y^2 + R_z^2\right]^{3/2}}$

and same for E_z

Thus $\quad \underline{E} = \dfrac{q\gamma \underline{R}}{4\pi\varepsilon_0\left(\gamma^2 R_x^2 + R_y^2 + R_z^2\right)^{3/2}}$

If θ is angle between $\underline{R}$ and x-axis then

$R_x = R\cos\theta \qquad R_y^2 + R_z^2 = R^2 - R_x^2 = R^2\sin^2\theta$

$\Rightarrow \gamma^2 R_x^2 + R_y^2 + R_z^2 = \gamma^2 R^2\cos^2\theta + R^2\sin^2\theta$

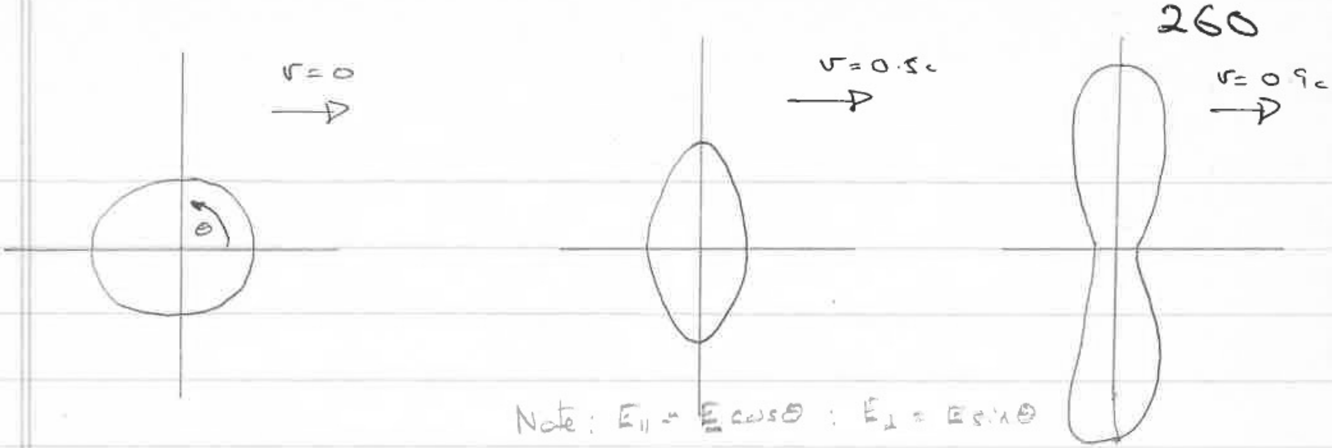
$$= R^2\left[\frac{\cos^2\theta + \sin^2\theta(1-\beta^2)}{(1-\beta^2)}\right]$$

$$= R^2\gamma^2\left(1 - \beta^2\sin^2\theta\right)$$

$$\Rightarrow \underline{E} = \frac{q\gamma\underline{R}}{4\pi\varepsilon_0 R^3\gamma^3\left(1-\beta^2\sin^2\theta\right)^{3/2}} = \underline{\frac{q\hat{\underline{R}}(1-\beta^2)}{4\pi\varepsilon_0 R^2\left(1-\beta^2\sin^2\theta\right)^{3/2}}}$$

This is the field due to a moving charge.

Note that the inverse square dependence on distance remains, but there is now a definite angular dependence.



Note: $E_{\parallel} = E\cos\theta$: $E_{\perp} = E\sin\theta$

e.g. $\theta = 0°$ (or 180°) $\quad E_{\parallel} = \dfrac{q(1-\beta^2)}{4\pi\varepsilon_0 R^2} \quad (E_{\perp} = 0)$

$\theta = 90° \qquad E_{\parallel} = 0 \qquad E_{\perp} = \dfrac{q}{R^2 4\pi\varepsilon_0 (1-\beta^2)^{1/2}} = \dfrac{q\gamma}{4\pi\varepsilon_0 R^2}$

Thus as β increases the shape of the field distribution changes from the sphere ($v=0$) becoming a spheroid flattened in the direction of $\underline{v}$ and finally becoming the double lobed shape above for $v \sim 0.9c$.

* $\underline{B}$ (in S) can most simply be found from

$$\underline{B} = \frac{\underline{v} \wedge \underline{E}}{c^2}$$

$\Rightarrow$ $\underline{B}$ is @ 90° to both $\underline{E}$ and $\underline{v}$

with $\underline{B} = \dfrac{\beta q (1-\beta^2)(\hat{\underline{x}} \wedge \hat{\underline{R}})}{c\, 4\pi\varepsilon_0 R^2 (1-\beta^2 \sin^2\theta)^{3/2}}$

$$\Rightarrow |\underline{B}| = \frac{q\beta(1-\beta^2)\sin\theta}{4\pi\varepsilon_0 c R^2 (1-\beta^2\sin^2\theta)^{3/2}}$$

e.g. $\theta = 0°$ and 180° $\quad |\underline{B}| = 0$

$\theta = 90° \qquad |\underline{B}| = \dfrac{q\beta}{4\pi\varepsilon_0 c R^2 (1-\beta^2)^{3/2}} = \dfrac{q\beta\gamma}{4\pi\varepsilon_0 R^2 c}$

$$= \frac{E_{\perp}\beta}{c}$$

$\underline{B}$ forms circles whose centres lie on the line of motion of the charge. But $\underline{B}$ is zero directly in front and directly behind the charge. [Imagine moving charge is a current. Lines of B form concentric circles around direction of motion of charge, but B = 0 directly in front + behind the charge]

* Finally we can investigate the potentials of a moving charge, starting from the static potential, in S'

$$\phi' = \frac{q}{4\pi\varepsilon_0 r'} \qquad \underline{A}' = 0$$

leading to (via the LT of the 4-vector $(\underline{A}, i\phi/c)$)

$\phi = \gamma(\phi' + VA'_x)$
$\phi = \gamma\phi'$

$$\phi = \frac{q}{4\pi\varepsilon_0 R\left(1-\beta^2\sin^2\theta\right)^{1/2}} \qquad (29\text{-}152)$$

(using same conversion of r' as with E)

$$\underline{A} = \frac{\mu_0 q \underline{v}}{4\pi\varepsilon_0 R\left(1-\beta^2\sin^2\theta\right)^{1/2}} \qquad (29\text{-}157)$$

As in the case of the $\underline{E}$ field, the potential is flattened in the direction of motion, as compared to the spherical potential when $v = 0$.

Also note that when $\beta \ll 1$ (small v)

$$\underline{A} = \frac{\mu_0 q \underline{v}}{4\pi\varepsilon_0 R}$$

exactly as expected in our original discussion of the vector potential.

* Chapter 29 : Problems – 3, 5, 7, 9, 13, 15, 17 → non EM relativity

21, 23, 25, 27, 29, 31 → EM and relativity

includes results from Ch 28.