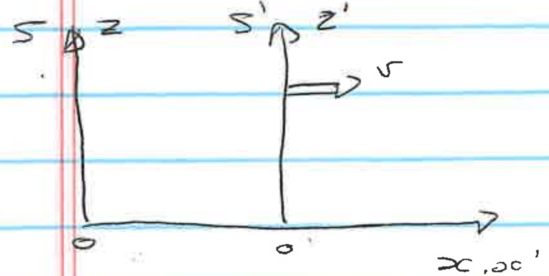
The Lorentz Transformation: A Derivation



Assume y, z components are unchanged and that x, t transformation are linear. [Due to homogeneity of space-time]

Then $\quad x = Ax' + Bt' \quad$ — (1)

$\quad t = Cx' + Dt' \quad$ — (2)

At origin of $S \quad x = 0$ and $x' = -vt'$

$$\Rightarrow \quad 0 = -Avt' + Bt'$$

$$\Rightarrow \quad B = +Av \quad \text{— (3)}$$

Consider an EM wave (flash) emitted at the origin when both origins coincide, then in the two reference frames we may write

$$x^2 + y^2 + z^2 = c^2t^2 \quad ; \quad x'^2 + y'^2 + z'^2 = c^2t'^2$$

where we assume the velocity of the EM wave is c in both reference frames.

Thus in the equation $\quad x^2 + y^2 + z^2 = c^2t^2$

using (1), (2) and (3)

$$\left(Ax' + (+Avt')\right)^2 + y'^2 + z'^2 = c^2\left(Cx' + Dt'\right)^2$$

$$A^2x'^2 + A^2v^2t'^2 + 2A^2vx't' + y'^2 + z'^2 = c^2C^2x'^2 + c^2D^2t'^2 + 2c^2CDx't'$$

$$x'^2\left(A^2 - c^2C^2\right) + y'^2 + z'^2 = c^2t'^2\left(-\frac{A^2v^2}{c^2} + D^2\right)$$

$$+ 2x't'\left(c^2CD - A^2v\right) \quad \text{— (4)}$$

But In $S' \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \quad$ — (5)

Comparing coefficients in (4) and (5)

$$A^2 - c^2C^2 = 1 \qquad x'^2 \qquad (6)$$

$$-A^2v^2/c^2 + D^2 = 1 \qquad t'^2 \qquad (7)$$

$$-A^2v + c^2CD = 0 \qquad x't' \qquad (8)$$

From (8) $A^2 = DCc^2/v$

Into (6) + (7)

$$\frac{DCc^2}{v} - c^2C^2 = 1 \quad - (9)$$

$$D^2 - \frac{\cancel{v^2}}{\cancel{c^2}}\frac{DC\cancel{c^2}}{\cancel{v}} = 1 \quad - (10)$$

From (10)

$$C = (D^2 - 1)/Dv$$

Into (9)

$$\frac{\cancel{D}c^2}{v}\frac{(D^2-1)}{\cancel{D}v} - \frac{c^2(D^2-1)^2}{D^2v^2} = 1$$

$$c^2D^2(D^2-1) - c^2(D^4 - 2D^2 + 1) = D^2v^2$$

$$-c^2D^2 + 2c^2D^2 - c^2 = D^2v^2$$

$$D^2c^2 - D^2v^2 = c^2$$

$$D^2 = \frac{c^2}{c^2(1 - v^2/c^2)} = \frac{1}{(1 - v^2/c^2)} = \gamma^2$$

$$D = \gamma$$

$$\Rightarrow C = \frac{(\gamma^2 - 1)}{\gamma v} = \left[\frac{1}{(1 - v^2/c^2)} - 1\right]\frac{1}{\gamma v}$$

$$= \left[\frac{1 - 1 + v^2/c^2}{1 - v^2/c^2}\right]\frac{1}{\gamma v}$$

$$C = \frac{v^2\gamma^2}{\gamma c^2 v} = \gamma v/c^2$$

$$\Rightarrow A^2 = \frac{DCc^2}{v} = \frac{\gamma^2 v c^2}{c^2 v} = \gamma^2 \Rightarrow A = \gamma$$

$$\Rightarrow B = Av = \gamma v$$

So that (1) and (2) become

$$x = \gamma\left(x' + vt'\right) = \gamma\left(x' + \beta c t'\right)$$
$$t = \gamma\left(t' + \frac{x'v}{c^2}\right) = \gamma\left(t' + \frac{\beta}{c}x'\right)$$

QED