22-2) ρ_F, J_F, P, M all given functions of (r,t)

General equations satisfied by potentials are

$$\nabla^2\phi + \frac{\partial}{\partial t}(\underline{\nabla}.\underline{A}) = -\frac{1}{\varepsilon_0}\left(\rho_f - \underline{\nabla}.\underline{P}\right) \quad \overset{(22-4)}{—\ (1)}$$

$$\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\left(\underline{J}_f + \underline{\nabla}\wedge\underline{M} + \frac{\partial P}{\partial t}\right) \quad \overset{(22-5)}{—\ (2)}$$

Lorentz condition for a vacuum is $(\sigma = 0,\ \varepsilon\to\varepsilon_0,\ \mu\to\mu_0)$

$$\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t} = 0$$

With the Lorentz condition above applied we can write (1) as

$$\nabla^2\phi - \mu_0\varepsilon_0\left(\partial^2\phi/\partial t^2\right) = -\frac{1}{\varepsilon_0}\left(\rho_f - \underline{\nabla}.\underline{P}\right) \quad — (1)$$

and (2) becomes

$$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\frac{\partial^2\underline{A}}{\partial t^2} = -\mu_0\left(\underline{J}_f + \underline{\nabla}\wedge\underline{M} + \frac{\partial P}{\partial t}\right) \quad -(2)$$

(1) is now a function of ϕ only (assuming ρ, P are known)

(2) is a function of $\underline{A}$ only ($\underline{J}_f$, $\underline{M}$, $\underline{P}$ known functions) thus the equations are separable.

22-5) Potential equations for l.i.h medium are given by

(22-9) (1) $$\underline{\nabla}^2 \underline{A} - \mu\sigma \frac{\partial \underline{A}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} + \mu\sigma\phi\right) = -\mu \underline{J}_f'$$

(22-8) (2) $$\nabla^2 \phi + \underline{\nabla}\cdot \frac{\partial \underline{A}}{\partial t} = -\frac{\rho_f}{\varepsilon}$$

In the Coulomb gauge $\underline{\nabla}\cdot\underline{A} = 0$

(2) becomes $\nabla^2\phi = -\rho_f/\varepsilon$

(1) becomes

$$\underline{\nabla}^2 \underline{A} - \mu\sigma \frac{\partial \underline{A}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}\left(\mu\varepsilon \frac{\partial \phi}{\partial t} + \mu\sigma\phi\right) = -\mu \underline{J}_f'$$

If the medium is also non-conducting $\sigma = 0$ and $\underline{J}_f = \underline{J}_f'$

(1) becomes

$$\underline{\nabla}^2 \underline{A} - \mu\varepsilon \frac{\partial^2 \underline{A}}{\partial t^2} = -\mu \underline{J}_f + \mu\varepsilon \frac{\partial}{\partial t}(\underline{\nabla}\phi)$$

22-7) $\rho_f = 0,\ \underline{J}_f = 0,\ \underline{M} = 0,\ \underline{P} = \underline{P}(\underline{r}, t)$

(22-4) $\nabla^2\phi + \frac{\partial}{\partial t}(\underline{\nabla}.\underline{A}) = -\frac{1}{\varepsilon_0}(\rho_f - \underline{\nabla}.\underline{P})$

(22-5) $\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\left(\underline{J}_f + \underline{\nabla}\wedge\underline{M} + \frac{\partial \underline{P}}{\partial t}\right)$

(a) with $\rho_f = \underline{J}_f = \underline{M} = 0$ we obtain

$$\underline{\nabla}^2\phi + \frac{\partial}{\partial t}(\underline{\nabla}.\underline{A}) = \underline{\nabla}.\underline{P}/\varepsilon_0 \quad — \ (1)$$

$$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}.\underline{A} + \mu_0\varepsilon_0\,\partial\phi/\partial t\right) = -\mu_0\frac{\partial \underline{P}}{\partial t} \quad -(2)$$

If $\underline{A} = \mu_0\varepsilon_0\left(\frac{\partial \underline{\Pi}}{\partial t}\right)$; $\phi = -\underline{\nabla}\cdot\underline{\Pi}$; $-\frac{\underline{P}}{\varepsilon_0} = \nabla^2\underline{\Pi} - \mu_0\varepsilon_0\frac{\partial^2\underline{\Pi}}{\partial t^2}$

then (1) becomes

$$-\underline{\nabla}.\underline{\nabla}(\underline{\nabla}.\Pi) + \frac{\partial}{\partial t}\left(\underline{\nabla}.\frac{\partial\Pi}{\partial t}\right)\mu_0\varepsilon_0 = \underline{\nabla}.\left[-\underline{\nabla}^2\underline{\Pi}_e + \mu_0\varepsilon_0\frac{\partial^2\Pi}{\partial t^2}\right]$$

$$-\underline{\nabla}.\underline{\nabla}(\underline{\nabla}.\underline{\Pi}) + \cancel{\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}.\underline{\Pi})} = \cancel{\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}.\underline{\Pi})} - \underline{\nabla}.\left[\underline{\nabla}^2\underline{\Pi}_e\right]$$

$$-\underline{\nabla}.\underline{\nabla}(\underline{\nabla}.\underline{\Pi}) = -\underline{\nabla}.\left[\underline{\nabla}(\underline{\nabla}.\underline{\Pi}) - \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\Pi})\right]$$

$$= -\underline{\nabla}.\underline{\nabla}(\underline{\nabla}.\underline{\Pi}) + \underbrace{\underline{\nabla}.\underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\Pi})}_{\text{div curl of any vector is zero}}$$

thus we have satisfied (1)

(2) becomes

$$\mu_0\varepsilon_0\frac{\partial}{\partial t}\left[\nabla^2\Pi\right] - \mu_0^2\varepsilon_0^2\frac{\partial^3\Pi}{\partial t^3}$$

$$-\underline{\nabla}\left(\underline{\nabla}.\frac{\partial\Pi}{\partial t}\right)\mu_0\varepsilon_0 + \mu_0\varepsilon_0\,\partial/\partial t\left[\underline{\nabla}(\underline{\nabla}.\Pi)\right] =$$

$$= +\mu_0\varepsilon_0\left(\frac{\partial \nabla^2\pi}{\partial t} - \mu_0\varepsilon_0\frac{\partial^3\pi}{\partial t^3}\right)$$

$\Rightarrow$ two sides of equation are equal, and we have satisfied (2)

(b) $$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \; ; \; \underline{B} = \underline{\nabla}\wedge\underline{A}$$

$$\Rightarrow \underline{E} = \underline{\nabla}(\underline{\nabla}\cdot\underline{\pi}) - \mu_0\varepsilon_0\frac{\partial^2\underline{\pi}}{\partial t^2}$$

where $\underline{\nabla}(\underline{\nabla}\cdot\underline{\pi}) = \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi}) + \nabla^2\underline{\pi}$

$$= \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi}) - \underline{P}/\varepsilon_0 + \mu_0\varepsilon_0\left(\partial^2\underline{\pi}/\partial t^2\right)$$

$\Rightarrow$

$$\underline{E} = \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi}) - \underline{P}/\varepsilon_0$$

and $$\underline{B} = \underline{\nabla}\wedge\underline{A} = \mu_0\varepsilon_0\underline{\nabla}\wedge\left(\partial\underline{\pi}/\partial t\right)$$

Alternatively

Start from

$$\nabla^2\underline{\pi} - \mu_0\varepsilon_0\frac{\partial^2\underline{\pi}}{\partial t^2} = -\frac{\underline{P}}{\varepsilon_0} \quad -(3)$$

take divergence of both sides of equation

$$\underline{\nabla}\cdot(\nabla^2\underline{\pi}) - \mu_0\varepsilon_0\frac{\partial}{\partial t}\left[\underline{\nabla}\cdot\frac{\partial\underline{\pi}}{\partial t}\right] = -\frac{\underline{\nabla}\cdot\underline{P}}{\varepsilon_0}$$

$$\underline{\nabla}\cdot\left[\underline{\nabla}(\underline{\nabla}\cdot\underline{\pi}) - \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi})\right] - \mu_0\varepsilon_0\frac{\partial}{\partial t}\left[\underline{\nabla}\cdot\frac{\partial\underline{\pi}}{\partial t}\right] = -\frac{\underline{\nabla}\cdot\underline{P}}{\varepsilon_0}$$

$$\nabla^2(-\underline{\nabla}\cdot\underline{\pi}) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left[\underline{\nabla}\cdot\frac{\partial\underline{\pi}}{\partial t}\right] = \frac{\underline{\nabla}\cdot\underline{P}}{\varepsilon_0}$$

But this is identical to (1) if $\phi = -\underline{\nabla} \cdot \underline{\Pi}$

and $\underline{A} = \mu_0 \varepsilon_0 \frac{\partial \pi}{\partial t}$

Taking the derivative w.r.t time of (3) we obtain

$$\nabla^2 \left(\frac{\partial \pi}{\partial t} \right) - \mu_0 \varepsilon_0 \frac{\partial^3 \pi}{\partial t^3} = - \frac{\partial \rho / \partial t}{\varepsilon_0}$$

$$\mu_0 \varepsilon_0 \nabla^2 \left(\frac{\partial \pi}{\partial t} \right) - \mu_0^2 \varepsilon_0^2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial \pi}{\partial t} \right) = - \mu_0 \, \partial \rho / \partial t$$

If $\phi = -\underline{\nabla} \cdot \underline{\Pi}$ and $\underline{A} = \mu_0 \varepsilon_0 \frac{\partial \underline{\Pi}}{\partial t}$ this becomes

$$\underline{\nabla}^2 \underline{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = - \mu_0 \frac{\partial \rho}{\partial t} \quad - (4)$$

but $\underline{\nabla} \cdot \underline{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{\Pi}) - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{\Pi})$

$= 0$

therefore we can add the gradient of this expression to the equation (4) giving

$$\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla} \left(\underline{\nabla} \cdot \underline{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t} \right) = - \mu_0 \frac{\partial \rho}{\partial t}$$

which is exactly equation (2).

22-8) $\rho_f = \underline{J}_f = \underline{P} = 0$

$$\Rightarrow \quad \nabla^2\phi + \frac{\partial}{\partial t}(\underline{\nabla}\cdot\underline{A}) = 0 \quad — \quad (1)$$

$$\underline{\nabla}^2\underline{A} - \mu_0\varepsilon_0\frac{\partial^2\underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0(\underline{\nabla}\wedge\underline{M})$$

$$— (2)$$

Using $\quad \nabla^2\pi - \mu_0\varepsilon_0\left(\frac{\partial^2\pi}{\partial t^2}\right) = -\mu_0 M$

taking curl of both sides

$$\underline{\nabla}\wedge\underline{\nabla}^2\pi - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}\wedge\underline{\pi}) = -\mu_0\underline{\nabla}\wedge\underline{M}$$

$$\underline{\nabla}\wedge\left(\underline{\nabla}(\nabla\cdot\pi) - \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi})\right) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}\wedge\underline{\pi}) = -\mu_0\underline{\nabla}\wedge\underline{M}$$

$$\underbrace{\underline{\nabla}\wedge\underline{\nabla}(\underline{\nabla}\cdot\pi)}_{\text{identically zero}} - \underline{\nabla}\wedge(\underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi})) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}\wedge\underline{\pi}) = -\mu_0\underline{\nabla}\wedge\underline{M}$$

$$-\underline{\nabla}(\underbrace{\underline{\nabla}\cdot(\underline{\nabla}\wedge\pi)}_{\text{identically zero}}) + \underline{\nabla}^2(\underline{\nabla}\wedge\underline{\pi}) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}\wedge\pi) = -\mu_0\underline{\nabla}\wedge\underline{M}$$

$$\underline{\nabla}^2(\underline{\nabla}\wedge\underline{\pi}) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}(\underline{\nabla}\wedge\underline{\pi}) = -\mu_0\underline{\nabla}\wedge\underline{M}$$

This is identically equal to (2) above <u>if</u>

$$\underline{A} = \underline{\nabla}\wedge\underline{\pi} \quad \underline{\text{and}} \quad \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right) = 0$$

If (1) is satisfied then $\quad \nabla^2\phi + \frac{\partial}{\partial t}(\underline{\nabla}\cdot\underline{A}) = 0$

The latter two conditions reduce to the following, when
$\underline{A} = \underline{\nabla} \wedge \underline{\pi}$

$$\underline{\nabla}\Big(\underbrace{\underline{\nabla}\cdot\underline{\nabla}\wedge\underline{\pi}}_{\text{zero}} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\Big) = 0$$

$$\nabla^2\phi + \frac{\partial}{\partial t}\Big[\underbrace{\underline{\nabla}\cdot(\underline{\nabla}\wedge\underline{\pi})}_{\text{zero}}\Big] = 0$$

or $\mu_0\varepsilon_0\dfrac{\partial(\underline{\nabla}\phi)}{\partial t} = 0 \Rightarrow \underline{\nabla}\phi$ independent of time.

and $\nabla^2\phi = 0$

Thus we conclude that $\underline{\underline{A} = \underline{\nabla}\wedge\underline{\pi}}$
and $\underline{\nabla^2\phi = 0}$ $\quad(\phi = \phi(\underline{r}))$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t} \qquad \underline{B} = \underline{\nabla}\wedge\underline{A}$$

$$\Rightarrow \underline{E} = \underline{-\underline{\nabla}\phi - \underline{\nabla}\wedge\frac{\partial\underline{\pi}}{\partial t}}$$

and $\underline{B} = \underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{\pi}) = \underline{-\nabla^2\underline{\pi} + \underline{\nabla}(\underline{\nabla}\cdot\underline{\pi})}$