24-3)
$$\psi_2 = \psi_{02} e^{i(kz+\omega t)} = \psi_{2a} e^{i\theta_2} e^{i(kz+\omega t)}$$

$$\psi_1 = \psi_{01} e^{i(kz-\omega t)} = \psi_{1a} e^{i\theta_1} e^{i(kz-\omega t)}$$

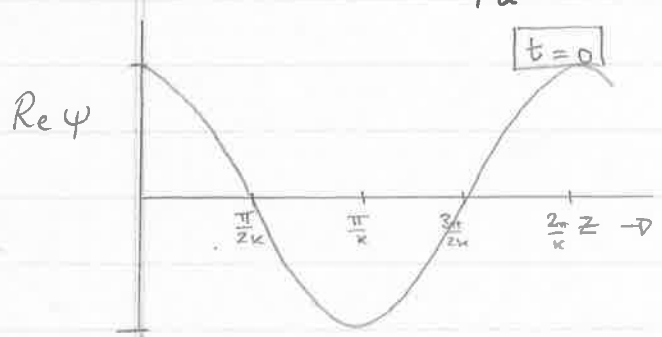
$\psi_{02} = \psi_{01}$ = real positive amplitudes $\Rightarrow$ $\psi_{2a} = U_{1a} = \psi_a$

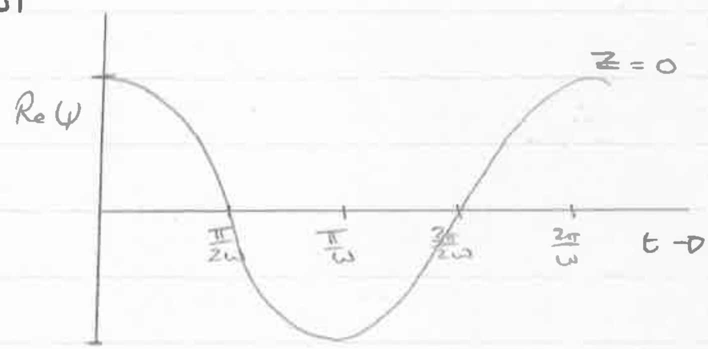
$\theta_1 = \theta_2 = 0$

$$\Rightarrow \quad \psi = \psi_2 + \psi_1$$
$$= \psi_a\left(\cos(kz-\omega t) + i\sin(kz-\omega t)\right)$$
$$= \qquad + \psi_a\left(\cos(kz+\omega t) + i\sin(kz+\omega t)\right)$$
$$= \psi_a\left(\cos(kz-\omega t) + \cos(kz+\omega t)\right)$$
$$+ i\psi_a\left(\sin(kz-\omega t) + \sin(kz+\omega t)\right)$$

$$\mathrm{Re}\,\psi = \psi_a\left[\cos kz\cos\omega t - \cancel{\sin kz\sin\omega t} + \cos kz\cos\omega t + \cancel{\sin kz\sin\omega t}\right]$$

$$= 2\psi_a \cos kz\cos\omega t$$

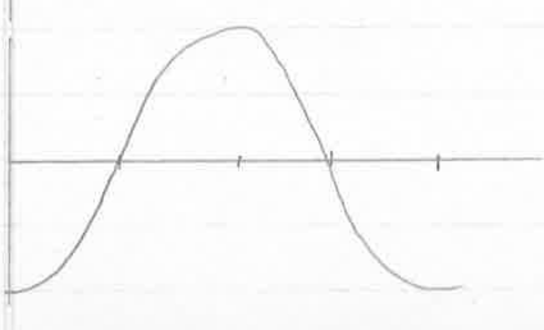


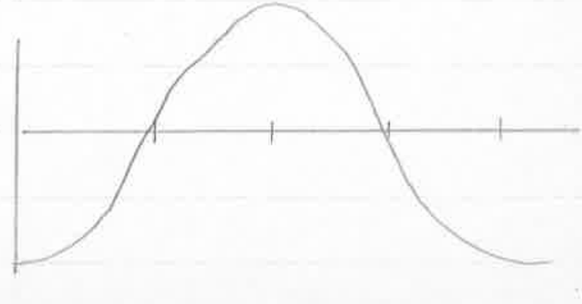

$t = \frac{\pi}{2\omega}$

$z = \frac{\pi}{2k}$

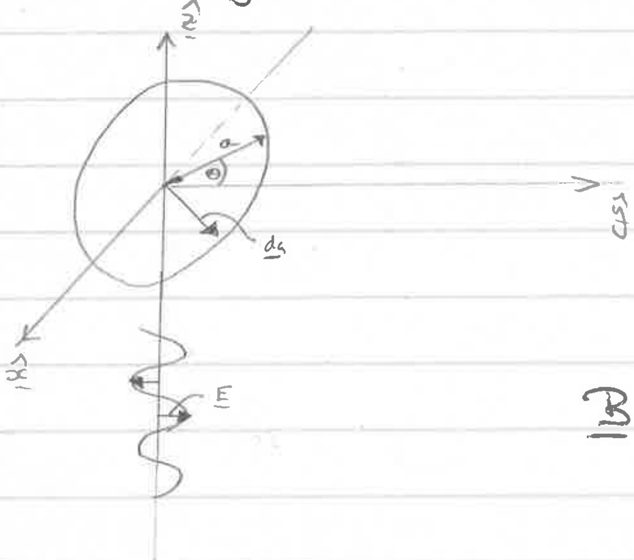
$t = \pi/k$



$z = \pi/k$



24-5) $\underline{E} = \hat{y} E_0 e^{i(kz-\omega t)}$




$$\underline{B} = \frac{k}{\omega}(\hat{z} \wedge \underline{E})$$

$$= \frac{k}{\omega}(\hat{z} \wedge \hat{y}) E_0 e^{ikz - i\omega t}$$

$$\underline{B} = -\hat{x} \frac{kE_0}{\omega} e^{i(kz-\omega t)}$$

$\underline{B}$ will create a magnetic flux through the loop

$$\Phi_B = \int_S \underline{B} \cdot \underline{da}$$

Where $\underline{da} = \pm(\hat{y} \sin\theta + \hat{x} \cos\theta)\, da$

$$\Phi_B = \int \cos\theta \frac{kE_0}{\omega} e^{i(kz-\omega t)}\, da$$

If we assume that $a \ll \lambda$, then the value of $\underline{B}$ will not vary much across the loop i.e. we can assume that $\underline{B}$ is equal to $\underline{B}(z=0)$ everywhere within the loop

$$\Rightarrow \Phi_B = \int \cos\theta \frac{kE_0}{\omega} e^{-i\omega t}\, da_x$$

$$\pi a^2 \cos\theta \frac{kE_0}{\omega} e^{-i\omega t}$$

24-5-2

Faradays Law $\qquad \mathcal{E} = -N \dfrac{d\Phi_B}{dt}$

$$|\mathcal{E}| = \frac{N\pi a^2 E_0 k}{\omega}(-i\omega)e^{-i\omega t}\cos\Theta$$

$$= N\pi a^2 E_0 k\, i(\cos\omega t - i\sin\omega t)\cos\Theta$$

$$= N\pi a^2 E_0 k \sin\omega t \cos\Theta$$

24-9) 1340 watts/$m^2$

the average value of the Poynting vector $\langle s \rangle$ is a measure of the solar constant, whose magnitude is given above.

$$\langle \underline{S} \rangle = \langle u \rangle \, \underline{v} \qquad \text{where} \quad \langle u \rangle = \langle u_m \rangle + \langle u_e \rangle$$

and $\underline{v}$ is the speed of light.

For a linearly polarised plane wave

$$E_x = E_0 \cos(kz - \omega t + \theta)$$
$$E_y = E_0 \cos(kz - \omega t + \theta)$$

with propagation is the z-direction.

We can write this as $\underline{E}_c = (\hat{x} + \hat{y}) E_0 e^{i(kz - \omega t + \theta)}$

where we take the real part of $\underline{E}_c$

$$\underline{E}_c = (\hat{x} + \hat{y}) E_0 \left[\cos(kz+\theta) + i \sin(kz+\theta)\right]\left[\cos\omega t - i \sin\omega t\right]$$
$$= (\hat{x} + \hat{y}) E_0 \left[\cos(kz+\theta)\cos\omega t + \sin(kz+\theta)\sin\omega t + i\left(\cos\omega t \sin(kz+\theta) - \sin\omega t \cos(kz+\theta)\right)\right]$$

$$\Rightarrow \quad \langle E^2 \rangle = 2 E_0^2 \cdot \tfrac{1}{2}.$$

now

$$\langle u_e \rangle = \langle u_m \rangle = \tfrac{1}{4} \, \varepsilon \langle E^2 \rangle = \tfrac{1}{4} \varepsilon |E_0^2|$$

$$\Rightarrow \quad \frac{\langle s \rangle}{v} = \frac{\langle s \rangle}{c} = \langle u \rangle = 2 \langle u_e \rangle = 2 \cdot \tfrac{1}{4} \varepsilon E_0^2$$

$$E_0^2 = \frac{2 \langle s \rangle}{c \cdot \varepsilon} = \frac{2 \times 1340}{3 \times 10^8 \times 8.85 \times 10^{-12}}$$

$$\Rightarrow E_0 = 1004 \text{ V/m}$$

Since $\langle u_e \rangle = \langle u_m \rangle$

24-9-2

and $\langle u_m \rangle = \frac{1}{4}\mu \langle H^2 \rangle$

$$= \frac{1}{4}\mu \frac{\langle B^2 \rangle}{\mu^2} = \frac{1}{4\mu} \langle B^2 \rangle$$

$$\frac{1}{4}\varepsilon E_0^2 = \frac{1}{4\mu} B_0^2$$

or $\frac{1}{4}\varepsilon E^2 = \frac{1}{4\mu} B^2$

$$B = E/c = \frac{1004}{3 \times 10^8} = 3.35 \times 10^{-6} \text{ Tesla.}$$

Non conductor

$$\underline{B} = \frac{\hat{z} \wedge \underline{E}}{c} \qquad c = \omega / k$$

Conductor

$$\underline{B} = \frac{(\hat{z} \wedge \underline{E})}{v} \cancel{k} = \frac{(\hat{z} \wedge \underline{E})}{\omega} k$$

but $k \Rightarrow |k| e^{i\Omega}$

$$\underline{B} = \frac{|k|}{\omega} e^{i\Omega} (\hat{z} \wedge \underline{E})$$

Non Conductor $\frac{|E|}{|B|} = c$

~~Non~~ conductor $\frac{|E|}{|B|} = \frac{\omega}{|k| e^{i\Omega}}$

$$= \frac{\omega}{(\alpha^2 + \beta^2)^{1/2}} e^{-i\Omega}$$

Thus $\frac{\langle u_e \rangle}{\langle u_B \rangle} = \mu \varepsilon \frac{E^2}{B^2}$

$$= \mu \varepsilon \underbrace{\frac{\omega^2}{(\alpha^2 + \beta^2)}}_{Q \text{ dependence.}}$$

$\Rightarrow \langle u_e \rangle \neq \langle u_B \rangle$ in conductors

For conductors $Q \ll 1$ $\alpha = \beta = \left(\frac{\mu \sigma \omega}{2}\right)^{1/2} \Rightarrow \langle u_e \rangle \ll \langle u_B \rangle$

non " $Q \gg 1$ $\beta = 0$ $\alpha = \omega \sqrt{\mu \varepsilon}$

$\Rightarrow \langle u_e \rangle = \langle u_B \rangle$

$$\frac{\langle u_e \rangle}{\langle u_B \rangle} = \frac{\mu \varepsilon \omega^2 \, 2}{2 \mu \sigma \omega} = \frac{\omega \varepsilon}{\sigma}$$

$$= Q \quad (\ll 1)$$

$\frac{\langle u_E \rangle}{\langle u_B \rangle} \ll 1$ or $\langle u_B \rangle \gg \langle u_e \rangle$

24-13) $\underline{E} = \hat{\underline{x}} E_\alpha e^{i(kz-\omega t+\Theta_\alpha)} + \hat{\underline{y}} E_\beta e^{i(kz-\omega t+\Theta_\beta)}$

$\underline{S} = \underline{E} \wedge \underline{H}$

where $\underline{E} = \text{Real}\left(E_0 e^{-i\omega t}\right)$

$= \text{Real}\left[\left(E_R + iE_I\right)\left(\cos\omega t - i\sin\omega t\right)\right]$

$= \underline{E}_R \cos\omega t + \underline{E}_I \sin\omega t$

Similarly $\underline{H} = \underline{H}_R \cos\omega t + \underline{H}_I \sin\omega t$

$\Rightarrow$

$$\underline{S} = \underline{E} \wedge \underline{H} = \left(\underline{E}_R \wedge \underline{H}_R\right)\cos^2\omega t + \left(\underline{E}_I \wedge \underline{H}_I\right)\sin^2\omega t + \left[\left(\underline{E}_R \wedge \underline{H}_I\right) + \left(\underline{E}_I \wedge \underline{H}_R\right)\right]\cos\omega t \sin\omega t$$

Averaging over time

$$\langle S \rangle = \tfrac{1}{2}\left[\left(\underline{E}_R \wedge \underline{H}_R\right) + \left(\underline{E}_I \wedge \underline{H}_I\right)\right]$$

Now

with $E_c = \left(E_R + E_I i\right) e^{-i\omega t}$

and $H_c = \left(H_R + H_I i\right) e^{-i\omega t}$

then $\underline{E}_c \wedge H_c^* = \left(\underline{E}_R + \underline{E}_I i\right) e^{-i\omega t} \wedge \left(\underline{H}_R - i\underline{H}_I\right) e^{i\omega t}$

$= \left(\underline{E}_R \wedge H_R + \underline{E}_I \wedge \underline{H}_I\right)$

$+ i\left(-\underline{E}_R \wedge H_I + \underline{E}_I \wedge \underline{H}_R\right)$

$\Rightarrow \quad \langle S \rangle = \tfrac{1}{2}\,\text{Real}\left(\underline{E}_c \wedge \underline{H}_c^*\right)$

but

$$\underline{H}_c = \left(\frac{\varepsilon}{\mu}\right)^{1/2} \hat{\underline{k}} \wedge \underline{E}_c = \left(\frac{\varepsilon}{\mu}\right)^{1/2} \left(\hat{\underline{z}} \wedge \underline{E}_c\right)$$

$$\langle S \rangle = \frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2} \text{Real}\left(\underline{E}_c \wedge \left(\hat{\underline{z}} \wedge E_c\right)^*\right)$$

$$= \frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2} \hat{\underline{z}} \left(\underline{E}_c \cdot \underline{E}_c^*\right)\text{Real}$$

$$= \frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2} \text{Re}\left[\left(\underline{E}_R + \underline{E}_I i\right) e^{-i\omega t}\right] \cdot \left[\left(\underline{E}_R - i\underline{E}_I\right) e^{i\omega t}\right] \hat{\underline{z}}$$

$$\langle S \rangle = \tfrac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^{1/2}\left[\underline{E}_R\cdot\underline{E}_R + \underline{E}_I\cdot\underline{E}_I\right]\hat{\underline{z}}$$

$$= \frac{1}{2Z}\left[\underline{E}_R\cdot\underline{E}_R + \underline{E}_I\cdot\underline{E}_I\right]\hat{\underline{z}}$$

Now in our case

$$\underline{E} = \hat{\underline{x}}\, E_\alpha\left[\cos(kz+\Theta_\alpha) + i\sin(kz+\Theta_\alpha)\right]e^{-i\omega t} + \hat{\underline{y}}\, E_\beta\left[\cos(kz+\Theta_\beta) + i\sin(kz+\Theta_\beta)\right]e^{-i\omega t}$$

$$\Rightarrow \quad \underline{E}_R = \hat{\underline{x}}\, E_\alpha \cos(kz+\Theta_\alpha) + \hat{\underline{y}}\, E_\beta \cos(kz+\Theta_\beta)$$

$$\underline{E}_I = \hat{\underline{x}}\, E_\alpha \sin(kz+\Theta_\alpha) + \hat{\underline{y}}\, E_\beta \sin(kz+\Theta_\beta)$$

$$\underline{E}_R\cdot\underline{E}_R = E_\alpha^2\cos^2(kz+\Theta_\alpha) + E_\beta^2\cos^2(kz+\Theta_\beta)$$

$$\underline{E}_I\cdot\underline{E}_I = E_\alpha^2\sin^2(kz+\Theta_\alpha) + E_\beta^2\sin^2(kz+\Theta_\beta)$$

and $$\langle S \rangle = \frac{(E_\alpha^2 + E_\beta^2)}{2Z}\hat{\underline{z}}$$

The Poynting vector (average) for a single component of $\underline{E}$ is found by setting, for example, the $\hat{y}$ component equal to zero

$$\Rightarrow \quad \langle S_x \rangle = \frac{E_\alpha^2}{2Z}$$

$$\Rightarrow \quad \langle S \rangle = \langle S_x \rangle + \langle S_y \rangle$$

24-15) $\underline{E} = \sum_k \underline{E}_{ok} \cos(kz - \omega_k t + \Theta_k)$

$$\underline{E} = \sum_k \text{Real}\left[E_{ok}\, e^{i(kz-\omega t+\Theta)}\right]$$

$$= \sum_k E_{ok}\, \text{Real}\left[(\cos(kz+\Theta) + i\sin(kz+\Theta))(\cos\omega t - i\sin\omega t)\right]$$

$$= \sum_k E_{ok} \left[\cos(kz+\Theta)\cos\omega t + \sin(kz+\Theta)\sin\omega t\right]$$

Now $\langle u_e \rangle = \frac{1}{2}\varepsilon \langle E^2 \rangle$
where

$$E^2 = \left[\sum_k E_{ok}\left[\cos(kz+\Theta)\cos\omega t + \sin(kz+\Theta)\sin\omega t\right]\right]^2$$

Each term in the summation must be squared giving

$$\sum_k E_{ok}^2 \left[\cos^2(kz+\Theta)\cos^2\omega t + \sin^2(kz+\Theta)\sin^2\omega t + 2\cos(kz+\Theta)\sin(kz+\Theta)\cos\omega t\sin\omega t\right]$$

Time averaging means the last term will dissappear leaving

$$\sum_k E_{ok}^2 \tfrac{1}{2}\left(\cos^2(kz+\Theta) + \sin^2(kz+\Theta)\right)$$

$$= \sum_k E_{ok}^2 \tfrac{1}{2}$$

But there will also be cross terms in the summation, a typical one will look like

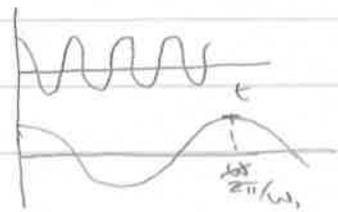
$$E_{ok_1}\cos(k_1 z + \Theta_{k_1})\cos\omega_{k_1} t \;\; E_{ok_2}\cos(k_2 z + \Theta_{k_2})\cos\omega_{k_2} t$$

Upon forming the time average the interesting part of this term is $\cos\omega_{k_1}t \, \cos\omega_{k_2}t$ .

The average value of this function is given by

$$\int_0^{2\pi/\omega_1} \cos\omega_1 t \, \cos\omega_2 t \; dt \Big/ \int_0^{2\pi/\omega_1} dt$$

where $\omega_2 > \omega_1$




$$\tfrac{1}{2}\int \left[\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t\right] dt$$

which averaged over an appropriate time interval is zero.
All cross terms will similarly vanish.
Leaving

$$\langle E^2 \rangle = \sum_k \tfrac{1}{2} E_{0k}^2$$

Therefore $\langle u_e \rangle = \tfrac{1}{2}\, \varepsilon \sum_k \tfrac{1}{2} E_{0k}^2$

which is exactly the sum of the densities of each component.

24-17) $\underline{k} = 314\,\hat{\underline{x}} + 314\,\hat{\underline{y}} + 444\,\hat{\underline{z}}$

This is the direction of propagation

In a vacuum $\nu = c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = \dfrac{\omega}{|\underline{k}|}$

$$\Rightarrow \quad \omega = 2\pi\nu = \frac{|\underline{k}|}{\sqrt{\mu_0 \varepsilon_0}}$$

(a) $\nu = \dfrac{|\underline{k}|}{2\pi\sqrt{\mu_0\varepsilon_0}}$ where $|\underline{k}| = 628 \ m^{-1}$

3 $= \dfrac{628 \times 3 \times 10^{8}}{2\pi} = 3 \times 10^{10}$ Hz

(b) $\lambda = c/\nu = 10^{-2}$ m

1

(c) Angle $\underline{k}$ makes with $\hat{\underline{x}}$ axis is given by

$$\hat{\underline{x}} \cdot \underline{k} = |\hat{\underline{x}}|\,|\underline{k}| \cos\theta_x$$

$$\cos\theta_x = \frac{\hat{\underline{x}} \cdot \underline{k}}{|\underline{k}|} = \frac{314}{628} \Rightarrow \theta_x = 60^\circ$$

6

Similarly $\cos\theta_y = \frac{1}{2} \Rightarrow \theta_y = 60^\circ$

$$\cos\theta_z = \frac{\hat{\underline{z}} \cdot \underline{k}}{|\underline{k}|} = \frac{444}{628} \Rightarrow \theta_z = 45^\circ.$$

24-19) $\underline{E}_+ = E_0(\hat{\underline{x}} - i\hat{\underline{y}})e^{i(kz-\omega t+\theta)}$

(a) We may write generally $E_x = E_1 \cos(kz - \omega t + \theta_1)$
$E_y = E_2 \cos(kz - \omega t + \theta_2)$

For circularly polarised light $|E_1| = |E_2| = E_0$
and $|\theta_1 - \theta_2| = \pi/2$

For r.h. circular polarisation $\theta_1 - \theta_2 > 0$
or $\theta_1 > \theta_2$
$\Rightarrow \theta_1 - \theta_2 = \pi/2$

and we can write
$$E_x = E_0 \cos(kz - \omega t + \theta_1)$$
$$E_y = E_0 \cos(kz - \omega t + \theta_1 - \pi/2)$$
$$= E_0 \cos(\pi/2 - (kz - \omega t + \theta_1))$$
$$= E_0 \sin(kz - \omega t + \theta_1)$$

$$\Rightarrow \underline{E} = E_0[\hat{\underline{x}} \cos(kz-\omega t+\theta_1) + \hat{\underline{y}} \sin(kz-\omega t+\theta_1)]$$

$$\underline{E} = \text{Real}[E_0 e^{i(kz-\omega t+\theta_1)}(\hat{\underline{x}} - i\hat{\underline{y}})]$$

[where the Real[ ] notation is often omitted since the actual $\underline{E}$ field is assumed always to be the real part]

(b) For left handed circularly polarised ~~light~~ wave
$\theta_1 - \theta_2 < 0$
$\theta_2 > \theta_1$ and $\theta_2 - \theta_1 = \pi/2$

$$\Rightarrow \underline{E} = E_0[\hat{\underline{x}} \cos(kz-\omega t+\theta_1) - \hat{\underline{y}} \sin(kz-\omega t+\theta_1)]$$
$$= \text{Real}[E_0 e^{i(kz-\omega t+\theta_1)}(\hat{\underline{x}} + i\hat{\underline{y}})]$$

24-31) $(\underline{E}_1, \underline{H}_1)$ $(\underline{E}_2\ \underline{H}_2)$ $\underline{J}_f' = 0$

Maxwells equations are

$$\nabla \cdot \underline{E} = \rho_f/\varepsilon \qquad \nabla \wedge \underline{E} = -\mu\, \partial \underline{H}/\partial t$$

$$\nabla \cdot \underline{H} = 0 \qquad \nabla \wedge \underline{H} = \sigma \underline{E} + \varepsilon\, \partial \underline{E}/\partial t$$

We must show that $\nabla \cdot [\underline{E}_1 \wedge \underline{H}_2 - \underline{E}_2 \wedge \underline{H}_1] = 0$

i.e.

$$\underline{H}_2 \cdot (\nabla \wedge \underline{E}_1) - \underline{E}_1 \cdot (\nabla \wedge \underline{H}_2) - \underline{H}_1 \cdot (\nabla \wedge \underline{E}_2) + \underline{E}_2 \cdot (\nabla \wedge \underline{H}_1) = 0$$

$$-\mu \underline{H}_2 \cdot \frac{\partial \underline{H}_1}{\partial t} - \underline{E}_1 \cdot \left[\sigma \underline{E}_2 + \varepsilon \frac{\partial \underline{E}_2}{\partial t}\right] + \mu \underline{H}_1 \cdot \frac{\partial \underline{H}_2}{\partial t} + \underline{E}_2 \cdot \left[\sigma \underline{E}_1 + \varepsilon \frac{\partial \underline{E}_1}{\partial t}\right] = 0$$

Now $\underline{H}_1 = H_{10} e^{-i\omega t}$

$$\frac{\partial \underline{H}_1}{\partial t} = -i\omega H_{10} e^{-i\omega t} = -i\omega \underline{H}_1$$

Similarly for $H_2$, $E_1$, $E_2$.

Therefore

$$+i\mu\omega \underline{H}_2 \cdot \underline{H}_1 - \cancel{\sigma \underline{E}_1 \cdot \underline{E}_2} + \cancel{i\varepsilon\omega \underline{E}_1 \cdot \underline{E}_2} - i\mu\omega \underline{H}_1 \cdot \underline{H}_2 + \cancel{\sigma \underline{E}_2 \cdot \underline{E}_1} - \cancel{i\varepsilon\omega \underline{E}_2 \cdot \underline{E}_1} = 0$$

QED