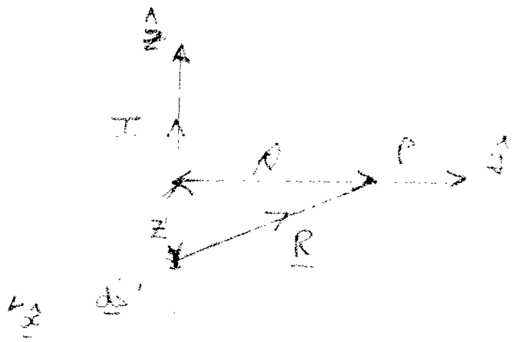
28-1)



$I = 0 \quad t < 0$

$I = I \quad t \geq 0$

$$\underline{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\underline{J}[\underline{r}', t - R/c]\, d\tau'}{R}$$

For a current element $I\,\underline{ds}'$ we replace $\underline{J}\, d\tau'$ by $I\,\underline{ds}'$ and $R = (\rho^2 + z'^2)^{1/2}$

(i) We know that at time $t < \rho/c$ there are no fields at P, since the smallest distance from P to the wire, ρ, is $> ct$.

$$\Rightarrow \quad t < \rho/c \quad \underline{B}, \underline{E}, \underline{A}, \underline{S} \text{ are all zero}$$

(ii) For $t \geq \rho/c$ $(\rho \leq ct)$ there is a disturbance at P due to the "sum" of all parts of the wire a distance $R \leq ct$ from P.

Thus at time t the potential $\underline{A}$ at P is caused by a wire of length $2[(ct)^2 - \rho^2]^{1/2}$

Using (16-30) $\quad \underline{A} = \hat{z}\,\frac{\mu_0 I}{4\pi}\int_{-L_1}^{L_2} \frac{dz'}{(\rho^2 + z'^2)^{1/2}} \qquad (\text{for } t \geq \rho/c)$

where $L_1 = [(ct)^2 - \rho^2]^{1/2}$

$L_2 = [(ct)^2 - \rho^2]^{1/2}$

thus $\underline{A} = \hat{z}\,\frac{\mu_0 I}{4\pi} \ln\left[\frac{(\cancel{\rho^2} + (ct)^2 - \cancel{\rho^2})^{1/2} + [(ct)^2 - \rho^2]^{1/2}}{(\cancel{\rho^2} + (ct)^2 - \cancel{\rho^2})^{1/2} - [(ct)^2 - \rho^2]^{1/2}}\right]$

$$= \hat{z}\,\frac{\mu_0 I}{4\pi} \ln\left[\frac{ct + ct\,(1 - (\rho/ct)^2)^{1/2}}{ct - ct\,(1 - (\rho/ct)^2)^{1/2}}\right]$$

$$= \hat{z}\,\frac{\mu_0 I}{4\pi} \ln\left(\frac{1 + F}{1 - F}\right)$$

Where $F = \left[1-(\rho/ct)^2\right]^{1/2}$

(iii) $\underline{B} = \underline{\nabla} \wedge \underline{A}$ (Where $A_\rho = A_\phi = 0$)

$$\Rightarrow \underline{B} = -\hat{\phi}\,\frac{\partial A_z}{\partial \rho}$$

$$= -\hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{\partial}{\partial \rho}\left[\ln(1+F) - \ln(1-F)\right]$$

$$= -\hat{\phi}\,\frac{\mu_0 I}{4\pi}\left[\left(\frac{1}{1+F}\right)\frac{\partial F}{\partial \rho} + \frac{1}{(1-F)}\frac{\partial F}{\partial \rho}\right]$$

where $$\partial F/\partial \rho = -\tfrac{1}{2}\left(1-(\rho/ct)^2\right)^{-1/2}\left(2\rho/(ct)^2\right)$$

$$= \frac{-\left(\rho/(ct)^2\right)}{\left[1-(\rho/ct)^2\right]^{1/2}} = \frac{-(\rho/ct)}{F}$$

$$\Rightarrow \underline{B} = +\hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{\left(\rho/(ct)^2\right)}{F}\,\frac{(1-F+1+F)}{(1+F)(1-F)}$$

$$= \hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{\left(\rho/(ct)^2\right)2}{F(1-F^2)} = \hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{\left(\rho/(ct)^2\right)2}{F\left[1-1+(\rho/ct)^2\right]}$$

$$= \hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{2}{F\rho} = \hat{\phi}\,\frac{\mu_0 I}{2\pi F\rho}$$

(iv) The Lorentz condition $\underline{\nabla}\cdot\underline{A} + \frac{1}{c^2}\,\partial\phi/\partial t = 0$

Where $\underline{\nabla}\cdot\underline{A} = 0$ (in this case) $\Rightarrow$ $\partial\phi/\partial t = 0$

or ϕ is not a function of time.

* In fact ϕ is independent of z & ϕ by symmetry, and independent of ρ since this would also imply a dependence on time.
$\Rightarrow \phi$ is constant and $\nabla\phi = 0$

$$\frac{\partial \underline{A}}{\partial t} = \hat{z}\,\frac{\mu_0 I}{4\pi}\,\frac{\partial}{\partial t}\left[\ln(1+F) - \ln(1-F)\right]$$

$$= \hat{z}\,\frac{\mu_0 I}{4\pi}\left[\left(\frac{1}{1+F}\right) + \frac{1}{(1-F)}\right]\frac{\partial F}{\partial t}$$

where $$\partial F/\partial t = \cancel{\tfrac{1}{2}}\left(1-(\rho/ct)^2\right)^{-1/2}(\rho/ct)^2\,\cancel{2}/t$$

$$= \frac{(\rho/ct)^2}{Ft}$$

$$\Rightarrow \quad \partial \underline{A}/\partial t = \hat{\underline{z}} \frac{\mu_0 I}{4\pi} \frac{2\,(\rho/ct)^2}{(1-F^2)Ft} = \hat{\underline{z}} \frac{\mu_0 I}{2\pi F t}$$

Now $\underline{E} = -\underline{\nabla}\phi - \partial\underline{A}/\partial t$

$$= -\hat{\underline{z}} \frac{\mu_0 I}{2\pi F t}$$

* Look at [illegible]

(v) $$\underline{S} = (\underline{E} \wedge \underline{B})/\mu_0 \qquad (= \underline{E} \wedge \underline{H})$$

$$= \frac{-(\hat{\underline{z}} \wedge \hat{\underline{\phi}})}{\mu_0} \frac{\mu_0^2 I^2}{4\pi^2 F^2 \rho t} = \hat{\underline{\rho}} \frac{\mu_0 I^2}{4\pi^2 F^2 \rho t}$$

(vi) As $t \to \infty$ $F \to 1$ and $\underline{A}$ must be evaluated as in (16-4) when the wire becomes infinitely long

(16-32) $$\underline{A} = \hat{\underline{z}} \frac{\mu_0 I}{2\pi} \ln\left[\frac{(4L_1L_2)^{1/2}}{\rho}\right] \qquad \left.\begin{matrix} L_1 \\ L_2 \end{matrix}\right\} \to \infty$$

As $t \to \infty$ $\underline{B} = \hat{\underline{\phi}} \frac{\mu_0 I}{2\pi\rho}$ exactly as expected from Ampere's Law for an infinite wire.

At $t \to \infty$ $\underline{E} \longrightarrow 0$ there is no longer a changing $\underline{B}$ field so that $\underline{E} = 0$ (static situation)

At $t \to \infty$ $S \to 0$ steady state no energy flow.

28.3)

Assume that P is in the radiation zone, then

$$\underline{E} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r}\sin\theta\, e^{i(kr-\omega t)}\,\hat{\underline{\theta}} \qquad (28\text{-}62)$$

and total radiated power $P = \dfrac{\mu_0 \omega^4 |p_0|^2}{12\pi c}$ (28-66)

with $P = 10^3$ watts.

In the equitorial plane $\theta = 90°$ so that the amplitude of $\underline{E}$ is given by

$$|\underline{E}| = \frac{k^2 p_0}{4\pi\varepsilon_0 r} = \frac{\omega^2 p_0}{4\pi\varepsilon_0 c^2 r}$$

$$\Rightarrow \quad |p_0|^2 = \frac{|\underline{E}|^2\, 16\pi^2 \varepsilon_0^2 c^4 r^2}{\omega^4}$$

Substituting in (28-66) we obtain

$$P = \frac{\mu_0 \cancel{\omega^4}}{{}_3\cancel{12\pi c}}\;\frac{|\underline{E}|^2\, \overset{4}{\cancel{16\pi^2}}\, \varepsilon_0^2\, \overset{3}{\cancel{c^4}}\, r^2}{\cancel{\omega^4}}$$

$$\Rightarrow \quad |\underline{E}|^2 = \frac{3P}{4r^2\varepsilon_0^2\mu_0 c^3\pi} = \frac{3P\cancel{c^2}}{4r^2\pi\varepsilon_0 \cancel{c^3}}$$

$$|\underline{E}| = \left(\frac{3P}{\pi\varepsilon_0 c}\right)^{1/2}\frac{1}{2r}$$

$$= \frac{(3\times10^3)^{1/2}}{(\pi\, 8.85\times10^{-12}\times 3\times10^8)^{1/2}}\;\frac{1}{2\times10^4}$$

$$= 0.03 \text{ V/m}.$$

28-4) The fraction $P_{\pm 10} / P_{TOT}$ is given by the ratio

$$\underbrace{\int_{\pm 10 \text{ deg}} \langle \underline{S} \rangle \cdot \underline{da}}_{\Downarrow} \Big/ \underbrace{\left(\frac{\mu_0 \omega^4 p_0^2}{12\pi c}\right)}_{\text{equation } (28-66)}$$

$$\frac{\mu_0 \omega^4 p_0^2}{32\pi^2 c} \int_0^{2\pi} \int_{\pi/2 - 10^\circ}^{\pi/2 + 10^\circ} \frac{\cancel{r^2} \sin^3\theta \, d\theta \, d\phi}{\cancel{r^2}}$$

$$\Downarrow$$

$$\frac{\mu_0 \omega^4 p_0^2}{16\pi c} \int_{\alpha_1}^{\alpha_2} \sin^3\theta \, d\theta \qquad \text{where } \alpha_1 = 80^\circ, \ \alpha_2 = 100^\circ$$

$$\Rightarrow \text{Fraction} = \frac{12}{16} \int_{\alpha_1}^{\alpha_2} (1 - \cos^2\theta) \sin\theta \, d\theta$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$

$$= \frac{3}{4} \int -(1-u^2) \, du$$

$$= -3/4 \left[u - u^3/3 \right]$$

$$= -3/4 \left[\cos\theta - (\cos^3\theta)/3 \right]_{\alpha_1}^{\alpha_2}$$

$$= -3/4 \left[\cos\alpha_2 - \cos\alpha_1 - 1/3 \left(\cos^3\alpha_2 - \cos^3\alpha_1 \right) \right]$$

$$= -3/4 \left[-0.3473 + 0.00349 \right]$$

$$= 0.258$$

i.e. About 25% of the total power is radiated within 10° of the equitorial plane.

28.5) (i) $$\underline{E} = -\frac{k^3 p_0}{4\pi\varepsilon_0}\left[\left(\frac{2i}{(kr)^2} - \frac{2}{(kr)^3}\right)\cos\theta\,\hat{\underline{r}} + \left(\frac{1}{(kr)} + \frac{i}{(kr)^2} - \frac{1}{(kr)^3}\right)\sin\theta\,\hat{\underline{\theta}}\right]e^{i(kr-\omega t)}$$

The physical $\underline{E}$ is the real part of $\underline{E}$
using $P = kr - \omega t$; $e^{iP} = \cos P + i\sin P$, thus the real part of $\underline{E}$ is given by

2 $$\underline{E}_R = -\frac{k^3 p_0}{4\pi\varepsilon_0}\left[\hat{\underline{r}}\left(-\frac{2\cos P}{(kr)^3} - \frac{2\sin P}{(kr)^2}\right)\cos\theta + \hat{\underline{\theta}}\left(\frac{\cos P}{(kr)} - \frac{\sin P}{(kr)^2} - \frac{\cos P}{(kr)^3}\right)\sin\theta\right]$$

(ii) $$\underline{B} = -\frac{\mu_0 k^2 \omega p_0}{4\pi}\left[\frac{1}{kr} + \frac{i}{(kr)^2}\right]\sin\theta\, e^{i(kr-\omega t)}\,\hat{\underline{\phi}}$$

2 Using $e^{i(kr-\omega t)} = \cos P + i\sin P$, the physical $\underline{B}$ $(=\underline{B}_R)$ is

$$\underline{B}_{Real} = -\hat{\underline{\phi}}\,\frac{\mu_0 k^2 \omega p_0}{4\pi}\sin\theta\left[\frac{\cos P}{kr} - \frac{\sin P}{(kr)^2}\right]$$

(iii) $\underline{S} = (\underline{E}\wedge\underline{B})/\mu_0$, where $\underline{E}$ and $\underline{B}$ are the physical fields

$\Rightarrow$

$$\underline{S} = \frac{1}{\mu_0}\frac{k^5 p_0^2 \omega \mu_0}{16\pi^2\varepsilon_0}\left[-\hat{\theta}\sin\theta\cos\theta\left(-\frac{2\cos^2 P}{(kr)^4} + \frac{2\sin^2 P}{(kr)^4} - \frac{2\sin P\cos P}{(kr)^3} + \frac{2\sin P\cos P}{(kr)^5}\right) + \hat{\underline{r}}\sin^2\theta\left(\frac{\cos^2 P}{(kr)^2} - \frac{\sin P\cos P}{(kr)^2} - \frac{\sin P\cos P}{(kr)^3} + \frac{\sin^2 P}{(kr)^4} - \frac{\cos^2 P}{(kr)^4} + \frac{\sin P\cos P}{(kr)^5}\right)\right]$$

3 $$= \frac{k^5 \omega p_0^2}{16\pi^2\varepsilon_0}\left[\hat{\underline{r}}\sin^2\theta\left(\frac{\cos^2 P}{(kr)^2} - \frac{\sin 2P}{(kr)^3} - \frac{\cos 2P}{(kr)^4} + \frac{\sin 2P}{2(kr)^5}\right) + \hat{\underline{\theta}}\sin\theta\cos\theta\left(\frac{\sin 2P}{(kr)^3} + \frac{2\cos 2P}{(kr)^4} - \frac{\sin 2P}{(kr)^5}\right)\right]$$

Using $\phi = (kr - \omega t)$ we can clearly see that in $\hat{r}$ and $\hat{\Theta}$ there are oscillatory components to $\underline{S}$

(iv) Taking the time average of $\underline{S}$ $\langle \sin 2\phi \rangle = \langle \cos 2\phi \rangle = 0$ and $\langle \cos^2 \phi \rangle = \frac{1}{2}$

3 $$\Rightarrow \langle \underline{S} \rangle = \frac{k^5 \omega p_0^2}{16\pi^2 \varepsilon_0} \frac{\sin^2\Theta}{(kr)^2} \cdot \frac{1}{2} \, \hat{r}$$

$$= \frac{\mu_0 \omega^4 p_0^2}{32\pi^2 r^2 c} \sin^2\Theta \, \hat{r} \qquad (28\text{-}65)$$

QED

28-6) For an oscillating electric dipole $\underline{p} = p_0 e^{-i\omega t}$

$$\Rightarrow \frac{d^2\underline{p}}{dt^2} = -\omega^2 \underline{p} \Rightarrow \left[\frac{d^2 p}{dt^2}\right] = -\omega^2 p_0 e^{-i\omega(t-r/c)}$$

Now for an electric dipole

$$\underline{B} = -\frac{\mu_0}{4\pi r}\frac{\omega^2}{c} \sin\theta \, p_0 e^{i(kr-\omega t)} \hat{\underline{\phi}}$$

From above we can write $\left[\frac{d^2\underline{p}}{dt^2}\right] = -\omega^2 p_0 e^{i(kr-\omega t)} \hat{\underline{z}}$

but $\hat{\underline{z}} \wedge \hat{\underline{r}} = \sin\theta \, \hat{\underline{\phi}}$

$$\Rightarrow \left[\frac{d^2\underline{p}}{dt^2}\right] \wedge \hat{\underline{r}} = -\omega^2 p_0 e^{i(kr-\omega t)} \sin\theta \, \hat{\underline{\phi}}$$

$$\Rightarrow \underline{B} = \frac{\mu_0}{4\pi r c}\left[\frac{d^2\underline{p}}{dt^2}\right] \wedge \hat{\underline{r}}$$

now $\underline{E} = c(\underline{B} \wedge \hat{\underline{r}})$

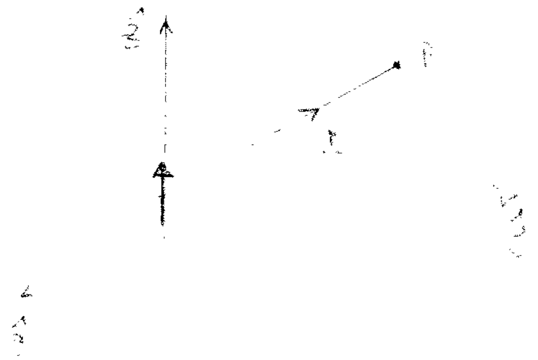
$$= \frac{\mu_0}{4\pi r}\left(\left[\frac{d^2\underline{p}}{dt^2}\right] \wedge \hat{\underline{r}}\right) \wedge \hat{\underline{r}}$$

QED

28-7) From (28-62) (28-63)

$$\underline{E} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r} \sin\Theta \, e^{i(kr-\omega t)} \, \hat{\underline{\Theta}}$$

$$\underline{B} = -\frac{\mu_0 k \omega p_0}{4\pi r} \sin\Theta \, e^{i(kr-\omega t)} \, \hat{\underline{\phi}}$$



$$\underline{ds}' = \hat{\underline{z}} \, ds'$$

and $\hat{\underline{z}} = \cos\Theta \, \hat{\underline{r}} - \sin\Theta \, \hat{\underline{\Theta}}$

thus

$$\underline{ds}' \wedge \hat{\underline{r}} = \sin\Theta \, \hat{\underline{\phi}} \, ds'$$

and

$$(\underline{ds}' \wedge \hat{\underline{r}}) \wedge \hat{\underline{r}} = \sin\Theta \, \hat{\underline{\Theta}} \, ds'$$

Now $\underline{p} = q \, \underline{ds}'$

and $\dfrac{d\underline{p}}{dt} = \dfrac{dq}{dt} \underline{ds}' = I' \underline{ds}'$

$$\Rightarrow \quad \frac{d^2\underline{p}}{dt^2} = \frac{dI'}{dt} \underline{ds}'$$

But $\underline{p} = \underline{p}_0 e^{-i\omega t} \Rightarrow d^2\underline{p}/dt^2 = -\underline{p}_0 e^{-i\omega t} \omega^2$

$$= \left(\frac{dI'}{dt}\right) \underline{ds}'$$

$$\Rightarrow \quad -\underline{p} \, \omega^2 = \left(\frac{dI'}{dt}\right) d\underline{s}'$$

$$\Rightarrow \quad p_0 e^{i(kr-\omega t)} = [p] = -\frac{ds'}{\omega^2} \left[\frac{dI'}{dt}\right]$$

both evaluated at the retarded time.

So that $$\underline{E} = \frac{k^2}{4\pi\varepsilon_0 r} \left[\frac{dI'}{dt}\right] \frac{\cancel{ds'}}{\omega^2} \frac{(\underline{ds}' \wedge \hat{\underline{r}}) \wedge \hat{\underline{r}}}{\cancel{ds'}}$$

$$= \frac{\mu_0}{4\pi r} \left[\frac{dI'}{dt}\right] (\underline{ds}' \wedge \hat{\underline{r}}) \wedge \hat{\underline{r}}$$

and $$\underline{B} = \frac{\mu_0 k \omega}{4\pi r}\left[\frac{dI'}{dt}\right]\frac{\cancel{ds'}}{\omega^2}\frac{(\underline{ds}' \wedge \hat{\underline{r}})}{\cancel{ds'}}$$

$$= \frac{\mu_0}{4\pi r c}\left[\frac{dI'}{dt}\right](\underline{ds}' \wedge \hat{\underline{r}})$$

28-8) Using the transformations

$$\underline{p} \to \frac{\underline{M}}{c} \qquad \underline{E} \to c\underline{B} \qquad \underline{B} \to -\underline{E}/c \qquad (28\text{-}77)$$

and the results of exercise (28-6) we find for a magnetic dipole

$$c\underline{B} = \frac{\mu_0}{4\pi r c}\left(\left[\frac{d^2\underline{M}}{dt^2}\right]\wedge\hat{\underline{r}}\right)\wedge\hat{\underline{r}}$$

$$\underline{B} = \frac{\mu_0}{4\pi r c^2}\left(\left[\frac{d^2\underline{M}}{dt^2}\right]\wedge\hat{\underline{r}}\right)\wedge\hat{\underline{r}}$$

and

$$-\frac{\underline{E}}{c} = \frac{\mu_0}{4\pi c^2 r}\left(\left[\frac{d^2\underline{M}}{dt^2}\right]\wedge\hat{\underline{r}}\right)$$

$$\underline{E} = -\frac{\mu_0}{4\pi c r}\left(\left[\frac{d^2\underline{M}}{dt^2}\right]\wedge\hat{\underline{r}}\right)$$

QED

28-9) (i) $$\underline{A}_{ed} = -\frac{i\mu_0 \omega p_0 e^{i(kr-\omega t)}}{4\pi r}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

now $$\frac{d\underline{A}_{ed}}{dt} = -i\omega\,\underline{A}_{ed} = -\frac{\mu_0\omega^2 p_0 e^{i(kr-\omega t)}}{4\pi r}\underbrace{\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)}_{\hat{z}}$$

In the radiation zone $$\underline{E}_{ed} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r}\sin\theta\, e^{i(kr-\omega t)}\,\hat{\theta}$$

$$= -\frac{\mu_0\omega^2}{4\pi r}[p]\sin\theta\,\hat{\theta}$$

evaluated at retarded time.

Now $$\left[\frac{dA}{dt}\right] = -\frac{\mu_0\omega^2}{4\pi r}[p]\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

3 and $$\left[\frac{dA}{dt}\right]\wedge\hat{r} = -\frac{\mu_0\omega^2}{4\pi r}[p]\sin\theta\,\hat{\phi}$$

$$\left(\left[\frac{dA}{dt}\right]\wedge\hat{r}\right)\wedge\hat{r} = -\frac{\mu_0\omega^2}{4\pi r}[p]\sin\theta\,\hat{\theta}$$

$$= \underline{E}_{ed}^{\text{rad zone}}$$

$$\underline{B}_{ed}^{rad} = -\frac{\mu_0 k\omega p_0}{4\pi r}\sin\theta\, e^{i(kr-\omega t)}\,\hat{\phi} = -\frac{\mu_0\omega^2[p]\sin\theta\,\hat{\phi}}{4\pi r c}$$

$$= \frac{1}{c}\left(\left[\frac{dA}{dt}\right]\wedge\hat{r}\right) \qquad \text{(from above)}$$

(ii) For magnetic dipole fields we have

$$\underline{A}_{md} = \frac{\mu_0 m_0}{4\pi r^2}(1-ikr)\,e^{i(kr-\omega t)}\sin\theta\,\hat{\phi}$$

7 $$\frac{d\underline{A}}{dt} = -i\omega\,\underline{A}_{md} = -\frac{\mu_0 m_0\omega\sin\theta\,(i+kr)\,e^{i(kr-\omega t)}}{4\pi r^2}\hat{\phi}$$

$$\left[\frac{dA}{dt}\right] = -\frac{\mu_0\omega[m]\sin\theta\,(i+kr)}{4\pi r^2}\hat{\phi} \underset{\text{in rad zone } (kr\gg 1)}{=} -\frac{\mu_0\omega k[m]\sin\theta\,\hat{\phi}}{4\pi r^2}$$

But $E^{rad\,zone}_{md} = \frac{\mu_0 \omega k [M] \sin\theta}{4\pi r} \hat{\phi}$

But $\left(\left[\frac{dA}{dt}\right] \times \hat{r}\right) \wedge \hat{r} = -\frac{\mu_0 \omega k [M] \sin\theta}{4\pi r^2} \hat{\phi} = E^{rad\,zone}_{md}$

and $\left(\left[\frac{dA}{dt}\right] \wedge \hat{r}\right) = -\frac{\mu_0 \omega [M]}{4\pi r^2} \sin\theta \, (i + kr) \, \hat{\theta}$

$= -\frac{kc [M]}{4\pi r^2 \varepsilon_0 c^2} \sin\theta \, (i + kr) \, \hat{\theta}$

In radiation zone $(kr \gg 1)$ $= \frac{-k^2 [M] \sin\theta}{4\pi r^2 \varepsilon_0 c} \hat{\theta} = B^{rad\,zone}_{md} c$

$\Rightarrow \quad B^{rad\,zone} = \frac{1}{c}\left(\left[\frac{dA}{dt}\right] \wedge \hat{r}\right)$

(iii) Finally for electric quadrupole fields

$$\underline{A}^{rad\,zone} = -\frac{\mu_0 \omega^2 Q_0}{48\pi c r} e^{i(kr - \omega t)} \left[(3\cos^2\theta - 1)\hat{r} - \tfrac{3}{2}\sin 2\theta \, \hat{\theta}\right]$$

$$\frac{d\underline{A}}{dt} = -i\omega \underline{A}^{rad\,zone}$$

4 $\quad \left[\frac{d\underline{A}}{dt}\right] \times \hat{r} = -i\omega [A^{rz}] \times \hat{r} = \frac{\mu i \omega^3}{48\pi c r} [Q_0] \tfrac{3}{2} \sin 2\theta \, \hat{\phi}$

$= \underline{B}^{rad\,zone}_{eq} c$

$$\left(\left[\frac{d\underline{A}}{dt}\right] \wedge \hat{r}\right) \wedge \hat{r} = \underline{B}^{rz} c \wedge \hat{r} = \underline{E}^{rz}$$

N.B. We could have used $\underline{E} = c\underline{B} \wedge \hat{r}$ throughout to show the $\underline{E}$ relationship, having proved the equality for $\underline{B}$

28-11) From the example on p.682 we obtain for a slowly moving accelerating charge

$$\underline{B} = \frac{q[a]\sin\theta}{4\pi\varepsilon_0 c^3 r}\,\hat{\underline{\phi}}$$

In this relationship θ is the angle between the direction of the dipole and $\hat{\underline{r}}$.

Assuming harmonic oscillation then $\underline{p} = \underline{p}_0 e^{-i\omega t}$

$$\underline{p} = q\underline{r}' \qquad \text{and} \qquad \frac{d^2\underline{p}}{dt^2} = q\,\underline{a}$$

$$q\underline{a} = -\omega^2\underline{p} \qquad \text{or} \qquad q[\underline{a}] = -\omega^2[\underline{p}]$$

that is $[\underline{a}]$ and $[\underline{p}]$ are in the same direction and

$$\hat{\underline{z}} \wedge \hat{\underline{r}} = (\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}) \wedge \hat{\underline{r}} = \hat{\underline{\phi}}\sin\theta$$

but $[\underline{p}]$, and therefore $[\underline{a}]$ are defined to be along $\hat{\underline{z}}$

$$\Rightarrow \qquad [\underline{a}] \wedge \hat{\underline{r}} = [a]\sin\theta\,\hat{\underline{\phi}}$$

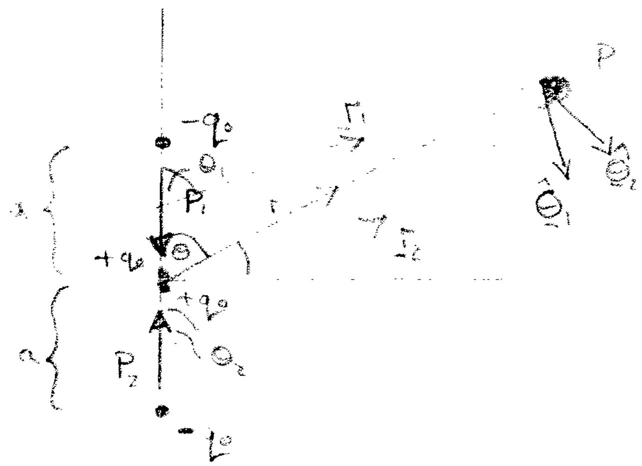
$$\Rightarrow \qquad \underline{B} = \frac{q[\underline{a}] \wedge \hat{\underline{r}}}{4\pi\varepsilon_0 c^3 r}$$

But $\quad \underline{E} = +(\underline{B} \wedge \hat{\underline{r}})\,c$

$$\Rightarrow$$

$$\underline{E} = \frac{q([\underline{a}] \wedge \hat{\underline{r}}) \wedge \hat{\underline{r}}}{4\pi\varepsilon_0 c^2 r}$$

28-12)



$$\underline{p}_1 = \sum_i q_i \underline{r}_i$$
$$\underline{p}_1 = -q_0 a \hat{\underline{z}}$$

$$\underline{p}_2 = +q_0 a \hat{\underline{z}}$$

Radiation zone $\underline{E}$ field due to an oscillating dipole is given by

$$\underline{E} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r} \sin\theta \, e^{i(kr-\omega t)} \hat{\underline{\theta}}$$

Thus due to p_1 we have

$$\underline{E}_1 = +\frac{k^2 p_1}{4\pi\varepsilon_0 r_1} \sin\theta_1 \, e^{i(kr_1-\omega t)} \hat{\underline{\theta}}_1$$

and for p_2 $\quad \underline{E}_2 = -\frac{k^2 p_2}{4\pi\varepsilon_0 r_2} \sin\theta_2 \, e^{i(kr_2-\omega t)} \hat{\underline{\theta}}_2$

Thus at P due to $\underline{E}$ fields $\underline{E}_1$ and $\underline{E}_2$

$$\underline{E} = \underline{E}_1 + \underline{E}_2$$
$$= +\frac{k^2 q_0}{4\pi\varepsilon_0}\left[\frac{a\sin\theta_1}{r_1}\hat{\underline{\theta}}_1 e^{i(kr_1-\omega t)} - \frac{a\sin\theta_2}{r_2}\hat{\underline{\theta}}_2 e^{i(kr_2-\omega t)}\right]$$

Now $\quad \underline{r}_1 = -\tfrac{1}{2}a\hat{\underline{z}} + \underline{r} \qquad \underline{r}_2 = \tfrac{1}{2}a\hat{\underline{z}} + \underline{r}$

$$r_1^2 = \tfrac{1}{4}a^2 + r^2 - a\hat{\underline{z}}\cdot\underline{r} \qquad r_2^2 = \tfrac{1}{4}a^2 + r^2 + a\hat{\underline{z}}\cdot\underline{r}$$
$$= \tfrac{1}{4}a^2 + r^2 - ar\cos\theta \qquad = \tfrac{1}{4}a^2 + r^2 + ar\cos\theta$$

$$\Rightarrow \quad \frac{1}{r_1} = \left(r^2 - ar\cos\theta + \tfrac{1}{4}a^2\right)^{-1/2}$$

$$= \tfrac{1}{r}\left(1 - \frac{ar\cos\theta}{r^2} + \tfrac{1}{4}\frac{a^2}{r^2}\right)^{-1/2}$$

If $a \ll r$

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a\cos\theta}{2r}\right) \quad \text{and} \quad \frac{1}{r_2} \simeq \frac{1}{r}\left(1 - \frac{a\cos\theta}{2r}\right)$$

Also for $a \ll r$ $\quad r_1 \simeq r - \frac{1}{2}a\cos\theta \simeq r\left(1 - \frac{a}{2r}\cos\theta\right)$

$\quad r_2 \simeq r + \frac{1}{2}a\cos\theta \simeq r\left(1 + \frac{a}{2r}\cos\theta\right)$

$$\Rightarrow \quad e^{i(kr_1 - \omega t)} = e^{i(kr - \omega t) + \frac{ka}{2}\cos\theta)}$$

$$= e^{i(kr-\omega t)}\, e^{-\frac{ika}{2}\cos\theta}$$

and $\quad e^{i(kr_2 - \omega t)} = e^{i(kr-\omega t)}\, e^{\frac{ika}{2}\cos\theta}$

$\Rightarrow$

$$\underline{E} = \frac{\omega^2 \mu_0 c^2}{4\pi c^2} q_0 a \left[\frac{\sin\theta_1}{r}\left(1 + \frac{a\cos\theta}{2r}\right) e^{-\frac{ika}{2}\cos\theta}\, \hat{\underline{\theta}}_1 - \frac{\sin\theta_2}{r}\left(1 - \frac{a\cos\theta}{2r}\right) e^{\frac{ika}{2}\cos\theta}\, \hat{\underline{\theta}}_2 \right] e^{i(kr-\omega t)}$$

Now $\quad e^{-\frac{ika}{2}\cos\theta} = 1 - \frac{ika\cos\theta}{2} - \frac{k^2a^2\cos^2\theta}{4 \cdot 2} + \cdots$

where $\quad ka \approx \frac{2\pi a}{\lambda}$ and all our assumptions in this chapter are $\lambda \gg a$

$\Rightarrow \quad ka \ll 1$ so that we can ignore terms in $(ka)^2$ and higher.

$$\Rightarrow \quad e^{-\frac{ika}{2}\cos\theta} \simeq 1 - \frac{ika\cos\theta}{2}$$

$$e^{\frac{ika}{2}\cos\theta} \simeq 1 + \frac{ika\cos\theta}{2}$$

$$\Rightarrow \quad \underline{E} = \frac{\omega^2 \mu_0 a q_0}{4\pi}\left[\frac{\sin\theta}{r} - \frac{ika\sin\theta\cos\theta}{2r} - \frac{\sin\theta}{r} - \frac{ika\cos\theta\sin\theta}{2r}\right] e^{i(kr-\omega t)}\, \hat{\underline{\theta}}$$

where we have assumed $\hat{\underline{\theta}}_1 \simeq \hat{\underline{\theta}}_2 \simeq \hat{\underline{\theta}}$

and $\sin\theta_1 \simeq \sin\theta_2 \simeq \sin\theta$

and we have ignored terms in $(1/r^2)$ and "higher"

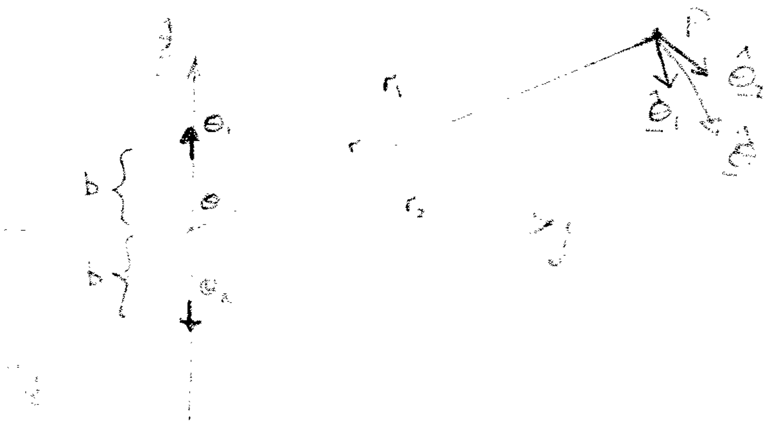
$$\Rightarrow \quad \underline{E} = \frac{-i\omega^2 \mu_0 a q_0 ka \sin 2\theta}{4\pi r 2}\, e^{i(kr-\omega t)}\, \hat{\underline{\theta}}$$

But $Q_0 = -4q_0a^2$ and $k = \omega/c$

Therefore

$$\underline{E} = \frac{i\mu_0\omega^3 Q_0}{32\pi rc} \sin 2\theta \, e^{i(kr-\omega t)} \, \hat{\underline{\theta}}$$

28-13)



Radiation zone field due to a magnetic dipole is given by

$$\underline{B} = -\frac{k^2 M_0}{4\pi\varepsilon_0 c^2 r} \sin\theta \, e^{i(kr-\omega t)} \, \underline{\hat{\theta}}$$

thus $\underline{B}_{TOT} = \underline{B}_1 + \underline{B}_2$ = sum of fields due to dipole ① and dipole ②

$$\underline{B}_{TOT} = -\frac{k^2 M_0}{4\pi\varepsilon_0 c^2}\left[\frac{\sin\theta_1}{r_1} e^{i(kr_1-\omega t)} \underline{\hat{\theta}}_1 - \frac{\sin\theta_2}{r_2} e^{i(kr_2-\omega t)} \underline{\hat{\theta}}_2\right]$$

Now using the same approximations as in exercise (28-12) with $\frac{1}{2}a \Rightarrow b$

$$\frac{1}{r_1} \simeq \frac{1}{r}\left(1 + \frac{b\cos\theta}{r}\right) \qquad \frac{1}{r_2} = \frac{1}{r}\left(1 - \frac{b\cos\theta}{r}\right)$$

and

$$e^{i(kr_1-\omega t)} = e^{i(kr-\omega t)} e^{-ikb\cos\theta}$$
$$\simeq e^{i(kr-\omega t)}\left[1 - ikb\cos\theta \; \ldots\right]$$
$$e^{i(kr_2-\omega t)} = e^{i(kr-\omega t)}\left[1 + ikb\cos\theta \; \ldots\right]$$

Where we have assumed $kb \ll 1$ and so ignored terms in $(kb)^2$ and higher powers.

Thus

$$\underline{B}_{TOT} = -\frac{k^2 M_0}{4\pi\varepsilon_0 c^2}\left[\frac{\sin\theta}{r}\left(1+\frac{b\cos\theta}{r}\right)\left(1-ikb\cos\theta\right) - \frac{\sin\theta}{r}\left(1-\frac{b\cos\theta}{r}\right) \times \left(1+ikb\cos\theta\right)\right] e^{i(kr-\omega t)} \underline{\hat{\theta}}$$

Where, as in (28-12) we have used $\sin\theta_1 \simeq \sin\theta_2 \simeq \sin\theta$ and $\underline{\hat{\theta}}_1 \simeq \underline{\hat{\theta}}_2 \simeq \underline{\hat{\theta}}$

$$\underline{B}_{TOT} = -\frac{M_0 \omega^2 \mu_0 \cancel{c^2}}{4\pi c^2 \cancel{c^2}}\left[-\frac{\sin\theta}{r}\, ikb\cos\theta - \frac{\sin\theta}{r}\, ikb\cos\theta\right] e^{i(kr-\omega t)}\, \hat{\underline{\theta}}$$

$$= \frac{i\mu_0 \omega^3}{4\pi c^3 r}\, b M_0 \sin 2\theta\, e^{i(kr-\omega t)}\, \hat{\underline{\theta}}$$

= Magnetic field due to magnetic quadrupole

Now $\quad \underline{E} = \left(\underline{B} \wedge \hat{\underline{r}}\right) c$

$$= -\frac{i\mu_0 \omega^3 b M_0}{4\pi c^2 r} \sin 2\theta\, e^{i(kr-\omega t)}\, \hat{\underline{\phi}}$$

28-16) (i) $\underline{E}_{TOT} = \underline{E}_{ed} + \underline{E}_{md}$

and $\underline{B}_{TOT} = \underline{B}_{ed} + \underline{B}_{md}$

Now

$$\langle \underline{S} \rangle = \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{TOT} \wedge \underline{B}^*_{TOT}\right)$$
$$= \tfrac{1}{2\mu_0}\,\mathrm{Real}\left((\underline{E}_{ed} + \underline{E}_{md}) \wedge (\underline{B}^*_{ed} + \underline{B}^*_{md})\right)$$
$$= \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{ed} \wedge \underline{B}^*_{ed} + \underline{E}_{md} \wedge \underline{B}^*_{md} + \underline{E}_{ed} \wedge \underline{B}^*_{md} + \underline{E}_{md} \wedge \underline{B}^*_{ed}\right)$$

(remember the * means take the complex conjugate)

$$\langle \underline{S} \rangle = \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{ed} \wedge \underline{B}^*_{ed}\right) + \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{md} \wedge \underline{B}^*_{md}\right) + \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{ed} \wedge \underline{B}^*_{md} + \underline{E}_{md} \wedge \underline{B}^*_{ed}\right)$$
$$= \langle \underline{S}_{ed} \rangle + \langle \underline{S}_{md} \rangle + \text{interference term}$$

But $\underline{E}_{ed} \wedge \underline{B}^*_{md} \propto \hat{\underline{\theta}} \wedge \hat{\underline{\theta}} = 0$

and $\underline{E}_{md} \wedge \underline{B}^*_{ed} \propto \hat{\underline{\phi}} \wedge \hat{\underline{\phi}} = 0$

$\Rightarrow$

$$\langle \underline{S} \rangle = \langle \underline{S}_{ed} \rangle + \langle \underline{S}_{md} \rangle$$

with no interference term

(ii) For the case of an electric dipole <u>and</u> linear electric quadrupole we make the substitutions $\underline{E}_{md} \rightarrow \underline{E}_{eq}$ and $\underline{B}_{md} \rightarrow \underline{B}_{eq}$

$$\Rightarrow \langle \underline{S} \rangle = \langle \underline{S}_{ed} \rangle + \langle \underline{S}_{eq} \rangle + \tfrac{1}{2\mu_0}\,\mathrm{Real}\left(\underline{E}_{ed} \wedge \underline{B}^*_{eq} + \underline{E}_{eq} \wedge \underline{B}^*_{ed}\right)$$

$$\left.\begin{array}{l}\text{Now } \underline{E}_{ed} \text{ is along } \hat{\underline{\theta}} \\ \quad \underline{B}_{eq} \text{ is along } \hat{\underline{\phi}}\end{array}\right\} \underline{E}_{ed} \wedge \underline{B}^*_{eq} \text{ is along } \hat{\underline{r}}$$

and $\underline{E}_{eq} \wedge \underline{B}^*_{ed}$ $(\hat{\underline{\theta}} \wedge \hat{\underline{\phi}})$ is along $\hat{\underline{r}}$

$\Rightarrow$ The interference term is <u>non zero</u>

28-17) From exercise (22-7) $\underline{A} = \mu_0 \varepsilon_0 \left(\frac{\partial \underline{\Pi}_e}{\partial t}\right)$

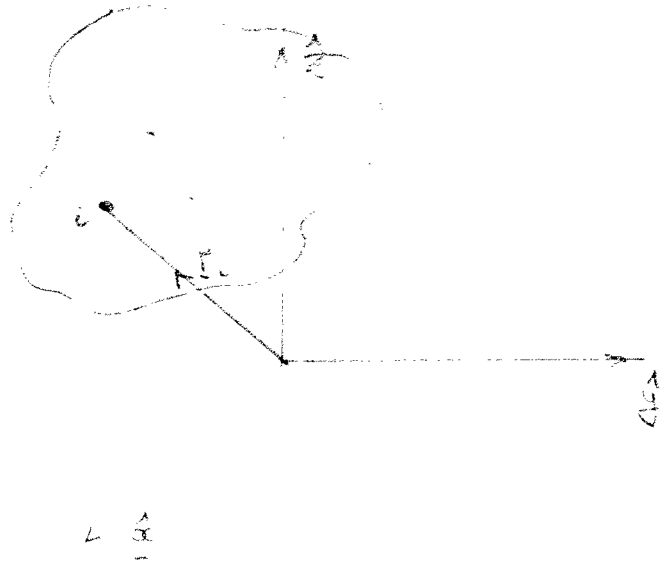
where $\underline{\Pi}_e = \frac{\underline{p}_0 e^{i(kr-\omega t)}}{4\pi\varepsilon_0 r}$

$$\Rightarrow \underline{A} = -\frac{i\mu_0 \cancel{\varepsilon_0}}{4\pi\cancel{\varepsilon_0} r} \underline{p}_0 e^{i(kr-\omega t)} \omega$$

$$= -\frac{i\mu_0 \omega}{4\pi r} p_0 e^{i(kr-\omega t)} \hat{\underline{z}} = \underline{A}_{ed} \quad (28\text{-}52)$$

QED

28-18)



(i) Definition of electric dipole moment

$$\underline{p} = \sum_{i=1}^{N} q_i \underline{r}_i$$

$\underline{R}_{cm}$ is given by

$$\underline{R}_{cm} = \sum_{i=1}^{N} m_i \underline{r}_i / M$$

Where M is total mass

$$\Rightarrow \underline{p} = \sum_{i=1}^{N} \frac{q_i m_i \underline{r}_i}{M_i}$$

$$= q/_M \sum_i m_i \underline{r}_i = (q/M) M \underline{R}_{cm}$$

Where (q/M) is the charge to mass ratio for every particle — assumed to be the same.

(ii) The vector potential for electric dipole radiation is given by

$$\underline{A}_{ed} = -\frac{i\mu_0 \omega \underline{p}_0 e^{i(kr-\omega t)}}{4\pi r} \qquad (28\text{-}41)$$

Assuming that the problem is referring to radiation in the radiation zone, we can obtain $\underline{E}$ and $\underline{B}$ from $\underline{A}_{ed}$

$$\underline{E} = \frac{\mu_0}{4\pi r}\left(\left[\frac{d^2\underline{p}}{dt^2}\right] \wedge \hat{\underline{r}}\right) \wedge \hat{\underline{r}} \qquad \underline{B} = \frac{\mu_0}{4\pi r c}\left[\frac{d^2\underline{p}}{dt^2}\right] \wedge \hat{\underline{r}}$$

But

$$\frac{d^2\underline{p}}{dt^2} = \left(\frac{q}{M}\right) M \frac{d^2\underline{R}_{cm}}{dt^2}$$

$$= \left(\frac{q}{M}\right) M \underline{a}_{cm} = \left(\frac{q}{M}\right) \underline{F}$$

Where $\underline{F}$ is the force on the system.

Thus both $\underline{E}$ and $\underline{B}$ are proportional to $[\underline{F}]$ (evaluated at the retarded time).

Hence unless there is an external force $\underline{F}$ on the system there will be no electric dipole radiation.

N.B. Note that in the "near" zone $\underline{E}, \underline{B}$ do not depend on d^2P/dt^2 (28-57)(28-59). But remember it is only the radiation zone terms that contribute to the total power radiated. (The other terms represent alternately incoming and outgoing waves, with no net radiation of energy on average).

(iii) For the magnetic dipole radiation in the radiation zone we have (from Exercise 28-8).

$$\underline{E} = -\frac{\mu_0}{4\pi c r}\left(\left[\frac{d^2\underline{M}}{dt^2}\right] \wedge \hat{\underline{r}}\right) \qquad \underline{B} = \frac{\mu_0}{4\pi c^2 r}\left(\left[\frac{d^2\underline{M}}{dt^2}\right] \times \hat{\underline{r}}\right) \wedge \hat{\underline{r}}$$

thus only if $\dfrac{d^2\underline{M}}{dt^2}$ is non-zero will we have magnetic dipole radiation.

Where do I go from here ... ?