29.3) $\tau_\mu = 2.22 \times 10^{-6}$ s "proper" lifetime.

(a) If meson has $\beta = 0.99$

Then lifetime is "dilated" to $\gamma \tau_\mu = \tau$

$$= \frac{\tau_\mu}{(1-\beta^2)^{1/2}}$$

$$\tau = \frac{2.22 \times 10^{-6}}{[1-(0.99)^2]^{1/2}} = 15.7 \times 10^{-6} \text{ s}$$

(b) Distance travelled is given by $d = \beta c \tau$

$$= 0.99 \times 3 \times 10^8 \times 15.7 \times 10^{-6}$$

$$= 4.67 \times 10^3 \text{ m}$$

[Alternatively in muon rest frame length is contracted and we may say $\underbrace{d'}_{\text{distance in } \mu \text{ rest frame}} = \tau_\mu \beta c$

$$\Rightarrow \quad d \text{ (in lab)} = d'\gamma = \tau_\mu \gamma \beta c$$

$$= \tau \beta c$$]

29-5) Single Lorentz Transformation leads to (along x-axis)

$$x' = \gamma(x - vt) \qquad t' = \gamma(t - vx/c^2)$$

with y and z co-ordinates unchanged

(a) Thus, two successive LT v_1 v_2 both along x-axis leads to

$$x'' = \gamma_2(x' - v_2 t') = \gamma_2(\gamma_1(x - v_1 t) - v_2\gamma_1(t - \tfrac{v_1 x}{c^2}))$$

$$t'' = \gamma_2(t' - v_2 x'/c^2) = \gamma_2(\gamma_1(t - v_1 x/c^2) - v_2\gamma_1(x - v_1 t)/c^2)$$

Where $\gamma_1 = \dfrac{1}{(1 - v_1^2/c^2)^{1/2}}$ $\quad \gamma_2 = \dfrac{1}{(1 - v_2^2/c^2)^{1/2}}$

therefore

$$x'' = (\gamma_2\gamma_1 + \gamma_2 v_2 \gamma_1 v_1/c^2)x - t(\gamma_2\gamma_1 v_1 + \gamma_2\gamma_1 v_2)$$

$$= \gamma_1\gamma_2(1 + v_2 v_1/c^2)\left[x - t\frac{(v_1 + v_2)}{(1 + v_2 v_1/c^2)}\right]$$

$$= \gamma_1\gamma_2(1 + v_1 v_2/c^2)[x - Vt]$$

where $V = (v_1 + v_2)(1 + v_2 v_1/c^2)^{-1}$

This will be a LT if $\gamma_1\gamma_2(1 + v_1 v_2/c^2) = \gamma_V$

$$\Rightarrow \gamma_1\gamma_2(1 + v_1 v_2/c^2) = \frac{1}{\left[1 - \dfrac{(v_1 + v_2)^2}{(1 + v_2 v_1/c^2)^2 c^2}\right]^{1/2}}$$

$$\frac{\cancel{(1 + v_1 v_2/c^2)}}{(1 - v_1^2/c^2)^{1/2}(1 - v_2^2/c^2)^{1/2}} = \frac{\cancel{(1 + v_1 v_2/c^2)}\, c}{[(1 + v_2 v_1/c^2)^2 c^2 - (v_1 + v_2)^2]^{1/2}}$$

$$\left[\left(1 + \frac{v_1^2 v_2^2}{c^4} + \frac{2v_1 v_2}{c^2}\right)c^2 - (v_1^2 + v_2^2 + 2v_1 v_2)\right]^{1/2} = \frac{(c^2 - v_1^2)^{1/2}(c^2 - v_2^2)^{1/2}}{c}$$

$$\left[c^2 + \frac{v_1^2 v_2^2}{c^2} + \cancel{2v_1 v_2} - v_1^2 - v_2^2 - \cancel{2v_1 v_2}\right]^{1/2}$$

$$[c^4 - c^2 v_1^2 - c^2 v_2^2 + v_1^2 v_2^2]^{1/2}/c$$

$$\left[c^4 + v_1^2 v_2^2 - c^2 v_1^2 - c^2 v_2^2\right]^{1/2} = \left[c^4 - c^2 v_1^2 - c^2 v_2^2 + v_1^2 v_2^2\right]^{1/2}$$

QED

(b) (29-37)
$$v_x = \frac{v_x' + V}{(1 + V v_x'/c^2)}$$

this is the velocity v_x in S in terms of the velocity v_x' in S' when S' moves @ V w.r.t S

For LT v_1

$$v_x = \frac{v_x' + v_1}{(1 + v_1 v_x'/c^2)}$$

thus after LT v_2

$$v_x = \frac{\dfrac{v_x'' + v_2}{(1 + v_2 v_x''/c^2)} + v_1}{\left[1 + \dfrac{v_1}{c^2}\dfrac{(v_x'' + v_2)}{(1 + v_2 v_x''/c^2)}\right]}$$

$$= \frac{(v_x'' + v_2 + v_1 + v_1 v_2 v_x''/c^2)\cancel{(1 + v_2 v_x''/c^2)}c^2}{(c^2 + v_2 v_x'' + v_1 v_x'' + v_1 v_2)\cancel{(1 + v_2 v_x''/c^2)}}$$

$$= \frac{v_x''\left[1 + v_1 v_2/c^2\right] + (v_1 + v_2)}{\left[\left(1 + \dfrac{v_1 v_2}{c^2}\right) + \dfrac{v_x''(v_1 + v_2)}{c^2}\right]}$$

$$= \frac{v_x'' + (v_1 + v_2)/(1 + v_1 v_2/c^2)}{1 + \dfrac{v_x''}{c^2}\dfrac{(v_1 + v_2)}{(1 + v_1 v_2/c^2)}}$$

But $V = (v_1 + v_2)/(1 + v_1 v_2/c^2)$

$\Rightarrow$

$$v_x = \frac{v_x'' + V}{\left(1 + \dfrac{v_x'' V}{c^2}\right)}$$ as expected.

29.7) $$\underline{a} = (a_x \; a_y \; a_z) = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt}\right)$$

Now $v_x = \frac{dx}{dt}$ $\quad v_y = \frac{dy}{dt}$ $\quad v_z = \frac{dz}{dt}$

where

$$x = \gamma(x' + Vt') \qquad y = y'$$
$$t = \gamma(t' + Vx'/c^2) \qquad z = z'$$

$$\Rightarrow \quad dx = \gamma(dx' + V dt')$$
$$dt = \gamma(dt' + V dx'/c^2)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{(dx' + V dt')}{(dt' + V dx'/c^2)} = \frac{v_x' + V}{(1 + V v_x'/c^2)} = v_x$$

Continue then by $a_x = \frac{dv_x}{dt}$

Similarly $$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + V dx'/c^2)} \; ; \quad \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + V dx'/c^2)}$$

$$\underline{v_y} = \frac{v_y'}{\gamma(1 + V v_x'/c^2)} \; ; \quad v_z = \frac{v_z'}{\gamma(1 + V v_x'/c^2)}$$

Now

$$a_x = \frac{dv_x}{dt} = \left[\frac{dv_x'}{(1 + V v_x'/c^2)} - \frac{(v_x' + V)\, V\, dv_x'/c^2}{(1 + V v_x'/c^2)^2}\right] \Big/ \gamma\left(dt' + \frac{V dx'}{c^2}\right)$$

$$= \frac{\left[dv_x'(1 + V v_x'/c^2) - (v_x' + V) V dv_x'/c^2\right]}{(1 + V v_x'/c^2)^2 \, \gamma(dt' + V dx'/c^2)}$$

$$= \frac{a_x'\left[1 + \cancel{V v_x'/c^2} - \cancel{V v_x'/c^2} - V^2/c^2\right]}{(1 + V v_x'/c^2)^2 \, \gamma(1 + V v_x'/c^2)}$$

Using $\eta = (1 + V v_x'/c^2)$

$$a_x = \frac{a_x'}{\gamma^3 \eta^3}$$

29-7-2

For $a_y = \dfrac{dv_y}{dt} = \dfrac{\left[\dfrac{dv_y'}{\gamma\left(1+Vv_x'/c^2\right)} - \dfrac{v_y'\gamma V\,dv_x'/c^2}{\gamma\left(1+Vv_x'/c^2\right)^2}\right]}{\gamma\left(dt' + V\,dx'/c^2\right)}$

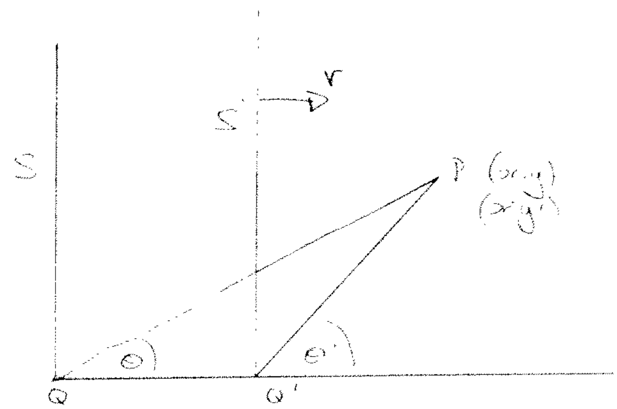
$$a_y = \frac{1}{\gamma\eta}\left[a_y' - \frac{v_y'\gamma V a_x'}{c^2\eta}\right] \Big/ \gamma\left(1+Vv_x'/c^2\right)$$

$$= \frac{a_y' - V v_y' a_x' \gamma / c^2\eta}{\gamma^2\eta^2}$$

$$= \left[a_y' - (V/c^2\eta)\,a_x' v_y'\right] / \gamma^2\eta^2$$

Similarly $a_z = \left[a_z' - (V/c^2\eta)\,a_x' v_z'\right] / \gamma^2\eta^2$

29-9)



θ, θ' small $\quad \beta \ll 1$

(a) Aberration formula

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'} \qquad (29\text{-}49)$$

$$(1+\beta\cos\theta')^{-1} = 1 - \beta\cos\theta' + \frac{(\beta\cos\theta')^2(-1)(-2)}{2!} + \dots$$

But for small θ', $\cos\theta' \sim 1$ and with β small we ignore all terms in β^2 leading to

$$\cos\theta \simeq (\cos\theta' + \beta)(1 - \beta\cos\theta')$$
$$\simeq \cos\theta' - \beta\cos^2\theta' + \beta - \beta^2\cos\theta'$$

Once again ignoring terms in β^2

$$\cos\theta \simeq \cos\theta' + \beta(1-\cos^2\theta')$$
$$\simeq \cos\theta' + \beta\sin^2\theta'$$

Now $\quad \cos\theta = (1-\sin^2\theta)^{1/2}$
$$= 1 - \tfrac{1}{2}\sin^2\theta + \frac{(\sin^2\theta)^2(\tfrac{1}{2})(-\tfrac{1}{2})}{2!} + \dots$$

But for small θ $\qquad \sin\theta \sim \theta \ll 1$

$$\Rightarrow \quad \cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$\Rightarrow \quad \cancel{1} - \frac{\Theta^2}{2} \simeq \cancel{1} - \frac{\Theta'^2}{2} + \beta\Theta'^2$$

$$\Theta^2 \simeq \Theta'^2 - 2\beta\Theta'^2$$

$$\Theta \simeq \Theta'(1-2\beta)^{1/2}$$

$$= \Theta'\left(1 - 2\beta \cdot \tfrac{1}{2} + \dots\right)$$

$$\Theta \simeq \Theta'(1-\beta)$$

QED

(b) Thus $\Theta - \Theta' \simeq -\beta\Theta'$

$$\Rightarrow \quad \frac{\Theta' - \Theta}{\Theta'} \simeq \beta = \frac{V}{c}$$

29-13) $B_\mu = \sum_\nu A_\nu T_{\nu\mu}$ $\qquad C_\mu = \sum_\nu T_{\mu\nu} A_\nu$

(a) If B_μ is a 4-vector then (→29-89)

$$B'_\mu = \sum_\nu A'_\nu T'_{\nu\mu} = \sum_\nu \left[\left(\sum_\rho a_{\nu\rho} A_\rho \right) \left(\sum_{\alpha\beta} a_{\nu\alpha} a_{\mu\beta} T_{\alpha\beta} \right) \right]$$

$$= \sum_\nu \sum_\rho \sum_{\alpha\beta} a_{\nu\rho} a_{\nu\alpha} a_{\mu\beta} A_\rho T_{\alpha\beta}$$

$$= \sum_\rho \sum_{\alpha\beta} \underbrace{\sum_\nu a_{\nu\rho} a_{\nu\alpha}}_{\delta_{\rho\alpha}} a_{\mu\beta} A_\rho T_{\alpha\beta}$$

$$= \sum_\rho \sum_{\alpha\beta} \delta_{\rho\alpha} a_{\mu\beta} A_\rho T_{\alpha\beta}$$

But the $\delta_{\rho\alpha}$ only is non zero when $\rho = \alpha$

$\Rightarrow$

$$B'_\mu = \sum_{\alpha\beta} a_{\mu\beta} A_\alpha T_{\alpha\beta}$$

$$B'_\mu = \sum_\beta a_{\mu\beta} B_\beta$$

which is exactly the transformation property of a 4-vector.

(b) For C_μ

$$C'_\mu = \sum_\nu T'_{\mu\nu} A'_\nu = \sum_\nu \left[\left(\sum_{\alpha\beta} a_{\mu\alpha} a_{\nu\beta} T_{\alpha\beta} \right) \left(\sum_\rho a_{\nu\rho} A_\rho \right) \right]$$

$$= \sum_\rho \sum_{\alpha\beta} \underbrace{\sum_\nu a_{\nu\beta} a_{\nu\rho}}_{\delta_{\beta\rho}} a_{\mu\alpha} T_{\alpha\beta} A_\rho$$

$$= \sum_\rho \sum_{\alpha\beta} \delta_{\beta\rho} a_{\mu\alpha} T_{\alpha\beta} A_\rho$$

$$C'_\mu = \sum_{\alpha\beta} a_{\mu\alpha} T_{\alpha\beta} A_\beta$$

$$= \sum_{\alpha} a_{\mu\alpha} C_\alpha$$

Which is exactly how a 4-vector transforms.

29-15) $k_\mu = (\underline{k}, i\omega/c)$

(b) If k_μ is a 4-vector then

$$k'_\mu = \sum_\nu a_{\mu\nu} k_\nu$$

Assuming relative motion only along the x axis we may use the values of $a_{\mu\nu}$ given in (29-77)

$$\Rightarrow \quad k'_1 = \gamma(k_1 + i\beta k_4)$$
$$k'_2 = k_2$$
$$k'_3 = k_3$$
$$k'_4 = \gamma(k_4 - i\beta k_1)$$

with $k_4 = i\omega/c$ we obtain

$$k'_x = \gamma(k_x - \omega\beta/c) \qquad k'_y = k_y$$
$$\omega' = \gamma(\omega - \beta k_x c) \qquad k'_z = k_z$$

But $k_x = |\underline{k}| \cos\Theta$

$$= (\omega \cos\Theta)/c$$

$$\Rightarrow \quad \omega' = \gamma(\omega - \beta c \omega \cos\Theta / c)$$
$$\omega' = \gamma\omega(1 - \beta\cos\Theta)$$

Which is the inverse transformation of (29-47) (Doppler Effect)

also

$$k'_x = \gamma(k_x - \omega\beta/c)$$
$$|\underline{k}'| \cos\Theta' = \gamma(|k|\cos\Theta - \omega\beta/c)$$
$$\frac{\omega'}{c}\cos\Theta' = \gamma\omega(\cos\Theta/c - \beta/c)$$

$$\omega' \cos\Theta' = \gamma\omega(\cos\Theta - \beta)$$

which is the inverse of the relationship below (29-48)

from which the Aberration formula (29-49) can be obtained.

(a) If k_μ is a 4-vector then $\sum_\mu k_\mu k_\mu$ is invariant

$\Rightarrow \quad \underline{k}^2 - \omega^2/c^2$ is invariant

But $|\underline{k}| = \omega/c$

$\Rightarrow \quad \underline{k}^2 - \omega^2/c^2 = 0$ <u>invariant</u>

29-17) $\quad F_\mu = \gamma \frac{d}{dt}(P_\mu) \qquad (29\text{-}102) + (29\text{-}99)$

(a) $\Rightarrow \quad F_1' = \gamma \frac{dP_1'}{dt'} = \frac{\gamma[\gamma\, dP_1 + i\beta\gamma\, dP_4]}{\gamma[dt - V\,dx/c^2]}$

$$= \frac{\gamma\, dP_1/dt + i\beta\gamma\, dP_4/dt}{[1 - V v_x/c^2]}$$

Using $\varepsilon = (1 - V v_x/c^2) = (1 - \beta v_x/c)$ we obtain

$$F_1' = \gamma[F_1/\gamma - \beta\,(dW/dt)/c]/\varepsilon$$

with $P_4 = iW/c$

But $F_1 = \gamma F_1 = \gamma f_x$

$$\Rightarrow \quad \cancel{\gamma} f_x' = \cancel{\gamma}[\cancel{\gamma} f_x/\cancel{\gamma} - \beta(dW/dt)/c]/\varepsilon$$

$$f_x' = \frac{f_x}{\varepsilon} - \frac{\beta}{c\varepsilon}\left(\frac{dW}{dt}\right)$$

but from (29-117) $\quad \frac{dW}{dt} = \mathbf{v}\cdot\mathbf{f} = v_x f_x + v_y f_y + v_z f_z$

$$\Rightarrow \quad f_x' = \left[f_x - \frac{\beta}{c}(v_x f_x + v_y f_y + v_z f_z)\right]\Big/\varepsilon$$

$$= \Big[f_x \underbrace{(1 - \beta v_x/c)}_{\varepsilon} - \beta(v_y f_y + v_z f_z)/c\Big]\Big/\varepsilon$$

$$f_x' = f_x - \frac{(v_y f_y + v_z f_z)V}{\varepsilon c^2}$$

(b) $\quad F_2' = \gamma \frac{dP_2'}{dt'} = \frac{\gamma\, dP_2}{\gamma[dt - V\,dx/c^2]}$

$$\gamma f_y' = \frac{dP_2/dt}{(1 - V v_x/c^2)} = \frac{F_2}{\gamma\varepsilon} = \frac{\gamma f_y}{\gamma\varepsilon}$$

$$\Rightarrow \quad f_y' = f_y/\gamma\varepsilon$$

(c) Similarly $f_z' = f_z / \gamma \varepsilon$

(d) $$F_4' = \frac{\gamma \, dP_4'}{dt'} = \frac{\gamma\left[\gamma dP_4 - i\beta\gamma \, dP_1\right]}{\gamma\left[dt - V\,dx/c^2\right]}$$

$$= \frac{\gamma \, dP_4/dt - i\beta\gamma \, dP_1/dt}{\left(1 - Vv_x/c^2\right)}$$

But $dP_1/dt = f_x$

and $dP_4/dt = \dfrac{i}{c}\dfrac{dW}{dt}$

$$\Rightarrow \quad \cancel{\gamma}\frac{\cancel{i}}{c}\left(\frac{dW}{dt}\right)' = \cancel{\gamma}\left[\frac{i}{c}\left(dW/dt\right) - i\beta f_x\right] / \varepsilon$$

$$\left(\frac{dW}{dt}\right)' = \left[\frac{dW}{dt} - V f_x\right] / \varepsilon$$